Qualifying Exam - Classical Mechanics - Fall, 2025

Problem 1. (40 pts) Rolling Cylinder with Internal Pendulum.

A hollow cylinder of mass M and radius R rolls without slipping down a plane inclined at an angle α to the horizontal. A simple pendulum, consisting of a point mass m and a massless rod of length ℓ ($\ell < R$), is suspended from the axis of the cylinder and is free to oscillate in the plane of motion. Let x be the distance the cylinder has rolled down the incline and θ be the angle the pendulum makes with the vertical.

(a) (25 pts) Derive the Lagrangian of the system in terms of the generalized coordinates x and θ and their time derivatives.

(b) (15 pts) From the Lagrangian, find the two equations of motion for the system.

Problem 2. (30 pts) Stability of a Central Force Orbit.

A particle of mass m moves under the influence of a central potential given by:

$$U(r) = -\frac{\alpha}{r^3} + \frac{\beta}{r^4}$$

where α and β are positive constants.

(a) (15 pts) For a given angular momentum L, show that a stable circular orbit exists and find its radius, r_0.

(b) (15 pts) If the particle is slightly perturbed from this stable circular orbit, determine the angular frequency of the small radial oscillations about r_0.

Problem 3. (30 pts) Relativistic Particle Production.

Consider the creation of a Higgs boson (H^0) from the collision of a very high-energy proton (p) with another proton that is initially at rest in the laboratory frame. The reaction is:

$$p + p \rightarrow p + p + H^0$$

Let m_p be the rest mass of the proton and M_H be the rest mass of the Higgs boson.

(a) (20 pts) By considering the system in the center-of-momentum frame, determine the minimum kinetic energy the incident proton must have in the laboratory frame for this reaction to occur. This is known as the threshold kinetic energy.

(b) (10 pts) Express your result from part (a) in the high-energy limit where $M_H \gg m_p$.

Electrodynamics Qualifying Exam

September 2025

You must provide the details or reasonings to justify your answers.

Problem 1: Simple Questions (5% each = 35% total)

1. Write down the four Maxwell's equations and specify your unit system.
2. Why are electromagnetic waves transverse in vacuum?
3. In one particular inertial frame, the electric and magnetic fields are measured as $\vec{E} = (9, -1, 7)$ and $c\,\vec{B} = (5, 0, -2)$ in some units, respectively. For the same field configuration can you measure in a different frame $\vec{E}' = (6, 0, 4)$ and $c\,\vec{B}' = (3, 1, -5)$?
4. Write down the Lorenz and Coulomb gauge conditions.
5. Write down a general definition for a Green's function.
6. Explain the terms Dirichlet and Neumann boundary conditions.
7. How many real degrees of freedom has a quadrupole moment? Explain your answer.

Problem 2: Simple derivations (10% each = 20% total)

1. Derive the conservation of charge from the Maxwell equations.
2. Derive the wave equation for electromagnetic fields in vacuum.

Problem 3: Moving charge (15%)

What are a point charge's electric and magnetic fields at a constant velocity $\vec{v}$?

Problem 4: Image Charges (18%)

Consider as volume the whole three-dimensional space. Centered at the origin is a grounded metal sphere with radius R. At $\vec{x} = (0, 0, d)$, $d > R$, we position a point charge with charge q. On the surface of the metal sphere and at $\vec{x} \to \infty$ the electric potential vanishes.

1. (5%) Find the electric potential outside the sphere and check that it fulfills the boundary conditions.
2. (8%) Calculate the quantity
$$\sigma = \epsilon_0 \, \hat{n} \cdot \vec{\nabla}\phi \; , \tag{1}$$
on the surface of the sphere S. Here $\hat{n}$ is the normal vector on the surface S pointing out of the considered volume. What is σ?
3. (5%) Calculate
$$Q = \int \sigma \, \mathrm{d}\, S \tag{2}$$
on the sphere. You can use Gauss' theorem to calculate Q but you need to define all the relevant volumes and surfaces and discuss what happens there.

Problem 5: Multipole Moments (12%)

Consider a homogeneously charged rectangular cuboid with charge Q and side lengths a, b, b with $a > b$. Choose the origin of the coordinate system to be in the center of the cuboid and put the axes along the symmetry axes of the object. The length of the cuboid in z-direction is a.

1. (3%) Write down an expression for the charge density, ϱ.
2. (4%) Calculate the spherical and the Cartesian monopole moment.
3. (5%) Calculate the spherical and the Cartesian dipole moments.

Remark: With spherical multipole moments we refer to the definition using spherical harmonics. With Cartesian multipole moments we refer to the definition using scalar, vector, tensor, . . ., in Cartesian coordinates. To calculate them you can use any coordinate system you prefer.

NTHU Physics Qualifying Exam
Quantum Mechanics 2025 Fall

Boldface characters like $\mathbf{v}$ refer to vectors ($= \vec{v}$). You may use the following formula:

- $\hbar = 1.05 \times 10^{-34}$ m^{2}kg/s, $c = 3 \times 10^8$ m/s, $e = 1.62 \times 10^{-19}$C, electron mass $m = 9.11 \times 10^{-31}$kg , proton mass $M = 1.67 \times 10^{-27}$kg,
- Pauli matrices:
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$
- Legendre Polynomials: $P_n(x)$, $-1 \le x \le 1$ is defined by the generating function,
$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n. \tag{2}$$
$$P_0(x) = 1, \quad P_1(x) = x. \tag{3}$$
and has the property:
$$\int_{-1}^{1} P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{mn}. \tag{4}$$
- For a 1-D simple harmonic oscillator (SHO), $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$, the raising and lowering operators are
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} (x - \frac{i}{m\omega} p), \quad a = \sqrt{\frac{m\omega}{2\hbar}} (x + \frac{i}{m\omega} p) \tag{5}$$
and $[a, a^\dagger] = 1$. The operators get their names from the facts that
$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a|n\rangle = \sqrt{n} |n-1\rangle, \tag{6}$$
where $|n\rangle$'s are the energy eigenstates of the 1D SHO.
- Schrodinger equation
$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle. \tag{7}$$
For a spherically symmetric potential $V(r)$, angular momentum is conserved and one can write the wave function as $\psi(\mathbf{r}) = \frac{R(r)}{r} Y_{lm}(\theta, \phi)$ and the (radial) time-independent Schrodinger equation is
$$-\frac{\hbar^2}{2m} \frac{d^2 R(r)}{dr^2} + \left(\frac{l(l+1)\hbar^2}{2mr^2} + V(r) \right) R(r) = E R(r). \tag{8}$$
- Time-dependent perturbation: For a perturbation H' that is turned on at time $t = 0$, $H = H_0 + H'$, the state $|\psi\rangle(t)$ can be expanded in terms of the unperturbed eigenstates $|n^{(0)}\rangle$:
$$|\psi(t)\rangle = \sum_{n=0} c_n(t) e^{-iE_n^{(0)} t/\hbar} |n^{(0)}\rangle. \tag{9}$$
If initially ($t = 0$) the particle is in the state $|n^{(0)}\rangle$, the transition amplitude for the particle to be in the state $|n'^{(0)}\rangle$ at time t is given by
$$c_{n'n} = \delta_{n'n} + \frac{1}{i\hbar} \int_0^t dt' e^{i\omega_{n'n} t'} H'_{n'n}, \tag{10}$$
where $\omega_{n'n} := (E_{n'}^{(0)} - E_n^{(0)})/\hbar$ and $H'_{n'n} := \langle n'^{(0)} | H' | n^{(0)} \rangle$.

- $3D$-Scattering: The wave function $\psi(r)$ of a particle scattering off a central potential $V(r)$ has the asymptotic form
$$\psi(\mathbf{r}) \to e^{ikz} + f(\theta)\frac{e^{ikr}}{r}, \quad r \to \infty, \tag{11}$$
where the incident wave is in the z-direction and $f(\theta)$ is called the scattering amplitude. The differential cross section is given by
$$d\sigma = |f(\theta)|^2 d\Omega. \tag{12}$$
and $d\sigma/d\Omega$ is called the differential cross section.

  It can be shown that $f(\theta)$ can be expressed as
$$f(\theta) = \frac{1}{k}\sum_l (2l+1)P_l(\cos\theta)e^{i\delta_l}\sin\delta_l \tag{13}$$
where δ_l is the phase shift of the l-th partial wave
$$\psi(\mathbf{r}) = \frac{1}{2ik}\sum_l (2l+1)P_l(\cos\theta)\left[\frac{e^{ikr+2i\delta_l}}{r} - \frac{e^{-i(kr-l\pi)}}{r}\right] \tag{14}$$
As a result, the total cross section is given by
$$\sigma = \sum_l \sigma_l, \tag{15}$$
where
$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l. \tag{16}$$

Examination Questions

Answer all questions.

You must show your work. No credits will be given if you don't show how you get your answers.

1. Physical quantities (12 points in total, 2 points each)
   Express each of the following quantities in terms of the fundamental constants $\hbar, e, c, m$ = electron mass, M = proton mass. Also give a rough estimate of numerical size for each.
   (a) Bohr radius (cm).
   (b) Binding energy of hydrogen
   (c) Compton wavelength of an electron (cm).
   (d) Classical electron radius (cm).
   (e) Electron rest energy (MeV).
   (f) Fine structure constant.

2. s-wave scattering (20 points)
   A 3D nonrelativistic particle of mass m and energy E scatters quantum-mechanically in a central potential $V(r)$ given by

   $$V(r) = \frac{\hbar^2}{2m}U(r), \quad U(r) = -2\left(\frac{\lambda}{\cosh \lambda r}\right)^2.$$

   Given that the differential equation

   $$\frac{d^2y}{dx^2} + k^2 y + \frac{2}{\cosh^2 x} y = 0$$

   has the solutions $y = e^{\pm ikx}(\tanh x \mp ik)$.
   (a) Determine the wave function for the s-partial wave.
   (b) Determine the s-wave contribution to the total cross section at energy E.

3. Angular momentum (18 points)
   A system of two particles each with spin 1/2 is described by an effective Hamiltonian

   $$H = A(s_{1z} + s_{2z}) + B\mathbf{s_1} \cdot \mathbf{s_2},$$

   where $\mathbf{s_1}, \mathbf{s_2}$ are the two spins, s_{1z}, s_{2z} are their z-components, and A, B are constants. Find all the eigenstates and energies of this Hamiltonian.

4. WKB approximation (15 points)
   (a) Derive the WKB approximation for the one-dimensional time-independent Schrodinger equation
   (b) Show that the approximation fails in the immediate neighborhood of a classical turning point.

5. Perturbation Theory (15 points)
   Consider a nucleon trapped inside a nucleus. The nucleon can be considered as a particle moving in a 3-dimensional spherically symmetric infinite potential well

$$V_0 = \begin{cases} 0, & r < R \\ \infty, & r > R. \end{cases}$$

   (a) Show that the ground state wave function is given by

$$\psi_0 = \sqrt{\frac{2}{R}} \frac{\sin(\pi r/R)}{r}$$

   (b) Suppose the nucleus get deformed in shape to a ellipsoid and the potential well becomes

$$V = \begin{cases} 0, & \frac{x^2+y^2}{b^2} + \frac{z^2}{a^2} < 1 \\ \infty, & \text{otherwise,} \end{cases}$$

   with $a = R/\sqrt{1+2\beta}, b = R/\sqrt{1-\beta}$, $\beta \ll 1$. Treat β as the perturbation parameter, show that the first order correction to the ground state energy vanishes, i.e.

$$\Delta E_0 = \langle \psi_0 | H' | \psi_0 \rangle = 0,$$

   where H' is the perturbation to the Hamiltonian.

6. Rabi's oscillation (20 points)
   Consider a two-level system with $E_1 < E_2$. The system is initially ($t = 0$) populated only at the lower energy level, that is,

$$c_1(0) = 1, \quad c_2(0) = 0.$$

   Suppose for $t \geq 0$, the system is subjected to a a time-dependent potential with the matrix elements:

$$V_{11} = V_{22} = 0, \quad V_{12} = \gamma e^{i\omega t}, \quad V_{21} = \gamma e^{-i\omega t} \quad (\gamma \text{ real})$$

   (a) Show by exactly solving the Schrodinger equation that for $t > 0$

$$|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \sin^2\left(\left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_{21})^2}{4}\right]^{1/2} t\right), \qquad \text{where } \omega_{21} := (E_2 - E_1)/\hbar.$$

   The periodic population and depopulation of the energy levels due to a periodic external field is called the Rabi's oscillation.

   (b) Use time-dependent perturbation theory to lowest nonvanishing order to compute the probability for the excited level 2 to be populated. When does the perturbative result fail?

   (c) Explain what physical effect is missing from the perturbative picture that leads to its failure.

Qualifying Examination – Statistical Mechanics

Sep 6-7, 2025

Please explain the logic behind your answers.

Problem 1. *Two-level system(10 points):* Consider a system of N distinguishable particles, which have two energy levels, $E_0 = -\mu B$ and $E_1 = \mu B$, for each particles. Here μ is the magnetic moment and B is the magnetic field. The particles populate the energy levels according to the classical distribution law.

1. Calculate the average energy of such system at temperature T
2. Calculate the specific heat of the system

Problem 2. *A simplified model of hemoglobin(10 points):* Hemoglobin is a protein in the blood that carries oxygen from the lungs to the muscles. It is formed by four units. Each unit can carry an O_2 molecule or not. For a hemoglobin, there will be 2^4 configurations as shown in Fig.1. (Each unit has two choices, carries nothing or an O_2 molecule. Therefore we have 2^4 possible choices.) As described in the caption of Fig.1, we can model the hemoglobin by assigning the binding energy $-\varepsilon_0$ and the additional gain, $-J$, if two nearby unit both bind with O_2(We try to pack O_2 to our hemoglobin as much as possible, a simple minded physicist guess). For example, considering the configuration in Fig. 2, the total energy of the simplified model is $(-\varepsilon_0) \times 2 + (-J) \times 1$. Consider we have $N/4$ hemoglobin, labeled by $\alpha = 1 \sim \frac{N}{4}$. On each hemoglobin, we have 4 units which can bind with O_2 molecules. We can use $\tau_{\alpha,i} = 0, 1$ for $i = 1 \sim 4$ to denote whether the i-th unit on the α-th hemoglobin is occupied by an O_2 molecule or not.

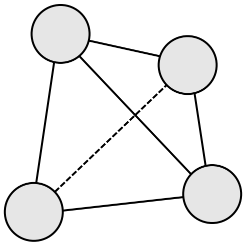

Figure 1: Schematic model of hemoglobin. The gray circles represent the unit that can carry an O_2 molecule. We assume the four units form a tetrahedral network structure. We can approximate the hemoglobin using a simplified model: The energy gain for an O_2 to bind with a unit is $-\varepsilon_0$. If two nearby units both contain O_2, the energy can further reduced by energy $-J$.

1. Write down the expression of the grand canonical partition function of the system with $\frac{N}{4}$ hemoglobin. Express the result using $x = exp\left[\frac{\varepsilon_0+\mu}{k_BT}\right]$ and $y = exp\left[\frac{J}{k_BT}\right]$. (Hint: we

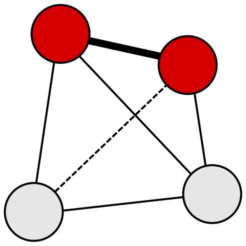

Figure 2: One particular configuration of hemoglobin. The red circles represent the units that are occupied by O_2 molecules. The thick line represent the additional energy reduction, $-J$, for nearby units are occupied by O_2 molecules. There will be other 5 configurations with equal energy.

consider the molecules of hemoglobin to be independent. So, we can factorize the partition function of the system into products of partition functions of individual hemoglobin molecule. Then, within each Hemoglobin, we can enumerate all the possible configurations and the corresponding energy to construct our partition function. Be careful about the number of configurations with equal energy. The degeneracy of energy in this system is not very regular, so one needs to specify them explicitly.)

2. Evaluate the average number $\langle M\rangle = \sum_{\alpha=1}^{N/4}\sum_{i=1}^{4}\langle\tau_{\alpha,i}\rangle$ of adsorbed molecules as a function of x, y.

Problem 3. *Free fermion problem(10 points):* Let the density of states of the electrons in a system be assumed to be a constant D for $\varepsilon > 0$ ($D = 0$ for $\varepsilon < 0$). Here, ε represents the energy. We assume the total number of electrons be equal to N. Here $N = D\int_0^\infty \frac{1}{e^{\beta(\varepsilon-\mu)}+1}d\varepsilon$.

1. Calculate the chemical potential μ at $T = 0K$.
2. When the system is in the regime of quantum degeneracy, estimate or derive the expression of the specific heat and show that it is proportional to T.

Problem 4. *Freely Jointed Chain (Gaussian model) and its mean field picture (40 points)*

Polymers are large(high molar mass) molecules composed of a large number of monomers bonded together to form a chain. In reality, the monomers are bounded covalently. That is, the potential energy between two monomers should be a complex function of the bonding angle between two nearby monomers. Let's make some assumptions to simplify our problem which leads to the freely jointed model. (We might over simplify the problem but let's just try it first to get a feeling about what kind of problem we are facing. Then we can ask how to put back more ingredients to make our analysis more realistic.)

- We assume the joint of the monomers can rotate freely, *i.e.* the potential energy is independent of the bonding angle, ϕ, between two nearby monomers.
- We assume the monomers can overlap with each other in space and have no interaction between each other.

The model with above two assumptions is the freely jointed chain model. It is the idealization of polymers analogous to ideal gas model for gases.

A key property that we are interested in is the size of the polymer. Usually the polymer coils up. The way they coils up will be a competition between the entropy and the energetics. Therefore,

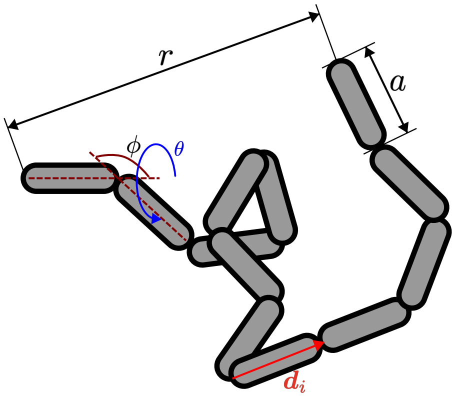

Figure 3: Schematic picture of a single chain polymer molecular formed by the monomers represented by the grey oval.

the size of the polymer is not simply the number of monomers, N, times the size of the monomer, a. Instead, we need to use some statistical description to characterize the size of the polymer. Usually, we use the root-mean-square end-to-end distance, $\langle r^2\rangle^{1/2}$, as a measure for the size of the polymer. Here, r is the distance from one end of the polymer to the other in three-dimensional space. The average is taken over all possible ways the polymer coils. One of the configuration is simple, if all the monomers are aligned in one direction, the end-to-end distance $r_{straight} = Na$. However, this is just one of the possible value of r, once the polymer coils, $\langle r^2\rangle^{1/2} < Na$.

1. It is interesting to observe that the freely jointed model in one-dimension is actually the simple one-dimensional random walk we discussed during the lecture. The step size is just the size of the monomer. The one-dimensional constraint restricts $\phi = 0/\pi$. Let's start in this simple limit and use x to represent the end-to-end distance in one-dimensional case. Derive the probability distribution $P_{1D}(x, N)$ when the number of monomers N is large.

2. For the three-dimensional case, we assume the orientation is completely random and the vector from monomer i and the vector for monomer $i+1$ are uncorrelated. *i.e.* $\langle \boldsymbol{d}_i \cdot \boldsymbol{d}_j\rangle_{i\neq j} = 0$. Therefore, we expect N to be distributed evenly $N_x = N_y = N_z = \frac{N}{3}$. Here, N_x is a rough definition of the monomer belongs to the monomer in x direction. We can simply consider the projection of the monomer to the x, y, z direction. If the projection to x direction is has the largest size, we said it is a x monomer that should be counted in N_x. Derive $\langle r^2\rangle^{\frac{1}{2}}$ and express it using N and a. This is a simple estimation for the size of the polymer. (Sometimes you will see people use the radius of gyration r_g^2 to estimate the size of the polymer. r_g is the average distance between monomers and the center of mass. It turns out the length scale, $\langle r_g^2\rangle$, will be proportional with $\langle r^2\rangle$. We will not discuss the calculation here, but just mention the fact that using the root mean square of the end-to-end distance capture the essential information for the size of the polymer.)

3. Now we have a rough idea about the freely jointed model. Let's assume we are in three-dimensional world. In the last problem, we should be able to have the probability distribution function $P_{3D}(r, N)$. Using the microcanonical ensemble, we should expect the probability distribution to related to the phase space volume as

$$P_{3D}(r, N) = \frac{\Omega(r, N)}{\int dr \Omega(r, N)} \equiv \frac{\Omega(r, N)}{A_N}. \tag{1}$$

Here, $\Omega(r, N)$ is the phase space volume for the size N polymer with the end-to-end distance r. A_N is the integral of $\Omega(r, N)$ over r. Use (1) to find the expression of the entropy, $S(r, N)$, for the size N polymer with the end-to-end distance r using N, a, r, k_B and A_N.

4. With the expression of entropy, we can derive the Helmholtz free energy of the system using $F(r, N) = U(r, N) - TS(r, N)$. Here, we are considering the freely jointed model, so there is no internal energy. The Helmholtz free energy is just $F(r, N) = -TS(r, N)$. Here is the interesting part, the free energy depends on r just like the spring with elastic energy ($E \propto k(\delta x)^2$). We can have the elasticity constant of the polymer. Find the expression of the elasticity constant k for the polymer described by the freely jointed model.

5. A rubber is an composite object with entangled polymers. They are difficult to anlayze during the qualify exam. However, from the simple calculation we achieved here we can explain some interesting physics. Consider a piece of rubber hanging a massive object below it. Suppose the rubber is strong enough such that the massive object just hang in the air. Next, if we heat up the rubber (hopefully we did not melt it.), what would you expect? Will the massive object be lifted higher? Or the other way around?

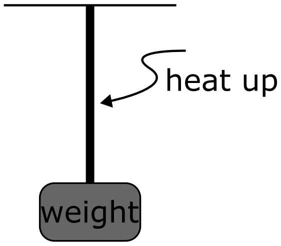

Figure 4: Schematic plot for the massive object hanging in the air.

6. Till now, we assume the non-interacting limit. So the free energy only consider the entropic effect. Now, we would like to extend the understanding to the case that the monomers can actually interact with each other. In order to proceed, we make the following assumptions:

   - In Fig. 3, we can clearly see the monomer is a rod like object which does not have rotational symmetry (it is not a sphere). However, **let's put that aside and assume the monomers are spherically symmetric objects with volume v that forbidden the overlap of two monomers.**
   - We can define the packing ratio, μ, of the monomers for the polymer as

     $$\mu = \frac{Nv}{r^3}. \tag{2}$$

     We drop the irrelevant coefficients just to get an estimation about the physics. It is the ratio of the total volume of the monomers and the volume given by the "size" of the polymer.

Since the interaction avoids two monomers to overlap in space. We can assume that the interaction between the monomers are pair-wise interaction. (They are just one monomer interacts with another one to satisfy the no-overlapping condition.) Estimate the interaction energy, $U_{int}(r, N, T)$, at temperature T for a polymer with length N at large N. (Hint: Think about the meaning of the packing ratio. How to estimate the pair-wise interactions using the packing ratio?)

7. Before we do any calculation, let's think about the effect of each term. The entropic contribution favors the value of small r to minimize the improved free energy. On the other hand, since the monomers repel each other due to the interaction energy, the $U_{int}(r, N, T)$ term prefers the value of large r in order to minimize the $F_{improved}(r, N, T)$. Therefore, the two effects competes with each other and fix the characteristic size of the polymer with length scale r^*. Use the above result to improve the free energy estimation by including the interaction between the monomers. We should have $F_{improved}(r, N, T) = U_{int}(r, N, T) + F_{entropic}(r, N, T)$. Find r^* as a function of v, a, N.
8. $r_{ideal} \sim N^{\frac{1}{2}}$, $r^* \sim N^{\alpha}$. Compare r_{ideal} and r^* at large N limit. Is $\alpha > \frac{1}{2}$, $\alpha < \frac{1}{2}$ or $\alpha = \frac{1}{2}$? Why is it a reasonable result?

Problem 5. *Bosons in Harmonic traps(30 points):* Let us consider a particle in the anisotropic harmonic-oscillator potential $V(\boldsymbol{r}) = \frac{1}{2}\left(K_x x^2 + K_y y^2 + K_z z^2\right)$. Therefore, we can consider the system as three independent harmonic oscillators in three different directions x, y and z. The corresponding energy levels can be parameterized by three non negative integers (n_x, n_y, n_z) as

$$E(n_x, n_y, n_z) = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z. \tag{3}$$

1. Let us consider n_i are continuous variables and neglect the zero-point energies (The $\frac{1}{2}\hbar\omega_i$ in $E(n_x, n_y, n_z)$.), we can simplify the energy as $E \approx \varepsilon_x + \varepsilon_y + \varepsilon_z$ where $\varepsilon_i = n_i\hbar\omega_i$. If we define $G(E)$ as the number of states below energy a specific energy E. Derive the expression of $G(E)$ in terms of E, $\omega_x, \omega_y, \omega_z$ and $\hbar$. (Hint: There are several way to derive this result. One approach is to consider the problem in the (n_x, n_y, n_z) space. What is the geometric meaning of the states below energy E?)
2. Another way to understand density of state is $g(E) = \frac{dG(E)}{dE}$. Use the above expression to get the density of states.
3. We can evaluate the particles in the excited states using

$$N_{ex} = \int_0^\infty dE g(E)\frac{1}{e^{E/k_BT} - 1}. \tag{4}$$

Here, we assume N_{ex} reaches its maximum for $\mu = 0$. The definition of the transition temperature, T_c, of Bose-Einstein condensation is when the total number of particles can be just accommodated in excited states. That is,

$$N = N_{ex}(T = T_c, \mu = 0) = \int_0^\infty dE g(E)\frac{1}{e^{E/k_BT} - 1}. \tag{5}$$

$\int_0^\infty dx \frac{x^{\alpha-1}}{e^x - 1} = \Gamma(\alpha)\zeta(\alpha)$. Here, $\zeta(\alpha) = \sum_{n=1}^\infty n^{-\alpha}$ is the Riemann zeta function and $\Gamma(\alpha)$ is the gamma function. Derive the transition temperature of this system and express the result using $N, \omega_x, \omega_y, \omega_z$ and the Riemann zeta function.