Electrodynamics Qualifying Exams

February 2025

You must provide the details or reasonings to justify your answers.

Problem 1: Simple Questions (5% each = 35% total)

1. Write down the four Maxwell's equations and specify your unit system.
2. Why are electromagnetic waves transverse in vacuum?
3. In one particular inertial frame, the electric and magnetic fields are measured as $\vec{E} = (9, -1, 7)$ and $c\,\vec{B} = (5, 0, -2)$ in some units, respectively. For the same field configuration can you measure in a different frame $\vec{E}' = (2, 0, 4)$ and $c\,\vec{B}' = (3, 5, 2)$?
4. Write down the Lorenz and Coulomb gauge conditions.
5. Write down a general definition for Green's function.
6. Explain the terms Dirichlet and Neumann boundary conditions.
7. How many real degrees of freedom has a quadrupole moment? Explain your answer.

Problem 2: Simple derivations (10% each = 20% total)

1. Derive the conservation of charge from the Maxwell equations.
2. Derive the wave equation for electromagnetic fields from the Maxwell equations.

Problem 3: Moving charge (15%)

What are a point charge's electric and magnetic fields at a constant velocity $\vec{v}$?

Problem 4: Image Charges (18%)

Consider two grounded metal plates at a right angle starting at the origin. They are aligned the x-axis and the y-axis and extend to $\pm\infty$ along the z-axis. On the surface of the metal plates and at infinity the potential fulfills the boundary conditions $\phi = 0$. In the volume there is a point charge q at $\vec{r}_1 = (d, d, 0)$.

1. (6%) Find the electric potential $\phi(\vec{r})$ in the volume V enclosed by the plates ($x > 0$, $y > 0$) and check that it fulfills the boundary conditions.

2. (5%) Calculate the surface charge density

$$\sigma = \epsilon_0 \, \hat{n} \cdot \vec{\nabla}\phi \, , \tag{1}$$

   with $\hat{n}$ the normal vector on the surface S_p pointing out of the volume.

3. (7%) Calculate the total charge

$$Q = \int \sigma \, \mathrm{d}\, S \tag{2}$$

   on the plates. If you use Gauss' theorem to calculate Q you need to choose a closed surface. Part of that surface must be the metal plates, S_p. How do you choose S_∞ and how much does it contribute to Q? S_∞ is the surface needed to get a closed surface. You can also directly evaluate the integral.

Problem 5: Multipole Moments (12%)

Consider a homogeneously charged rectangular cuboid with charge Q and side lengths a, b, b with $a > b$. Choose the origin of the coordinate system to be in the center of the cuboid and put the axes along the symmetry axes of the object. The length of the cuboid in z-direction is a.

1. (3%) Write down an expression for the charge density, ϱ.

2. (4%) Calculate the spherical and the Cartesian monopole moment.

3. (5%) Calculate the spherical and the Cartesian dipole moments.

Qualifying Exam - Classical Mechanics - Spring, 2025

Problem 1. (20 pts) General questions.

(a) (10 pts) For an oscillator that obeys $\ddot{x}+x+x^5=0$, how does its oscillation period compare to 2π? Please provide reasoning to justify your answer.

(b) (10 pts) Consider the following reaction,

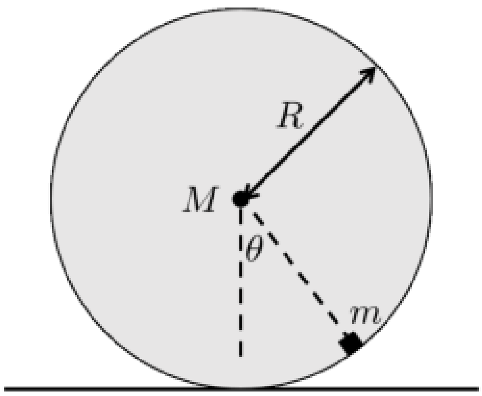
$$\gamma + p \rightarrow p + \pi^0,$$

where a proton (p) is at rest in the laboratory frame, and it is collided with a photon (γ). It produces a pion (π^0) after the collision. **Find the laboratory threshold photon energy** (in terms of rest masses m_p and m_π) for this reaction to happen.

Problem 2. (40 pts) A mass m is fixed to a given point on the rim of a wheel of radius R that rolls without slipping on the ground. Assume the wheel is massless, except for a mass M locates at its center. The motion of the system can be described by the dynamics of the angle (θ) between the vertical and the adjoining line connecting m and M, see figure below.

(a) (20 pts) Employ the generalized coordinate θ to write down the Lagrangian of the system.

(b) (20 pts) Obtain the equation of motion in the case of small oscillations ($\theta \ll 1$). What is the frequency of small oscillations?

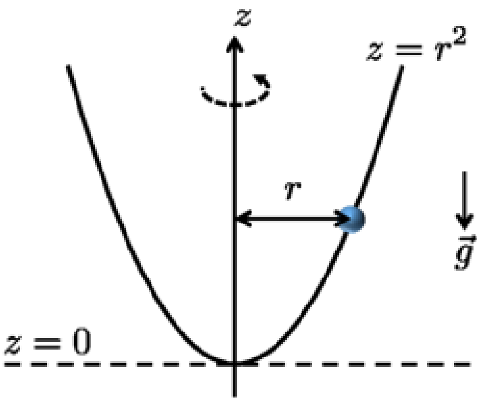


Problem 3. (40 pts) A particle of mass m subject to gravity slides along a frictionless parabolic wire (of negligible mass) $z = r^2$ and the parabolic wire also rotates around the z-axis freely, as in the figure shown below.

(a) (5 pts) Write down conserved quantities of this system if any.

(b) (20 pts) If the particle is exhibiting a uniform circular motion of a fixed radius r_0, determined the stable radius r_0.

(c) (15 pts) What is the oscillation frequency if the particle is perturbed around the equilibrium position r_0 along the r-direction?


z
$z = r^2$
r
$\vec{g}$
$z = 0$

NTHU Physics Qualifying Exam
Quantum Mechanics 2025 Spring

Boldface characters like $\mathbf{v}$ refer to vectors ($= \vec{v}$). You may use the following formula:

- $\hbar = 1.05 \times 10^{-34}$ m^{2}kg/s, $c = 3 \times 10^8$ m/s, $e = 1.62 \times 10^{-19}$C, electron mass $m = 9.11 \times 10^{-31}$kg , proton mass $M = 1.67 \times 10^{-27}$kg,

- Pauli matrices:
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

- Legedre Polynomials: $P_n(x)$, $-1 \leq x \leq 1$ is defined by the generating function,
$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n. \tag{2}$$
$$P_0(x) = 1, \quad P_1(x) = x. \tag{3}$$
and has the property:
$$\int_{-1}^{1} P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{mn}. \tag{4}$$

- For a 1-D simple harmonic oscillator (SHO), $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$, the raising and lowering operators are
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} (x - \frac{i}{m\omega} p), \quad a = \sqrt{\frac{m\omega}{2\hbar}} (x + \frac{i}{m\omega} p) \tag{5}$$
and $[a, a^\dagger] = 1$. The operators get their names from the facts that
$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a|n\rangle = \sqrt{n} |n-1\rangle, \tag{6}$$
where $|n\rangle$'s are the energy eigenstates of the 1D SHO.

- Schrodinger equation
$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H|\psi\rangle. \tag{7}$$
For a spherically symmetric potential $V(r)$, angular momentum is conserved and one can write the wave function as $\psi(\mathbf{r}) = \frac{R(r)}{r} Y_{lm}(\theta, \phi)$ and the (radial) time-indepedent Schrödinger equation is
$$-\frac{\hbar^2}{2m} \frac{d^2 R(r)}{dr^2} + \left(\frac{l(l+1)\hbar^2}{2mr^2} + V(r) \right) R(r) = ER(r). \tag{8}$$

- Time-dependent perturbation: For a perturbation H' that is turned on at time $t = 0$, $H = H_0 + H'$, the state $|\psi\rangle(t)$ can be expanded in terms of the unperturbed eigenstates $|n^{(0)}\rangle$:
$$|\psi(t)\rangle = \sum_{n=0} c_n(t) e^{-iE_n^{(0)} t/\hbar} |n^{(0)}\rangle. \tag{9}$$
If initially ($t = 0$) the particle is in the state $|n^{(0)}\rangle$, the transition amplitude for the particle to be in the state $|n'^{(0)}\rangle$ at time t is given by
$$c_{n'n} = \delta_{n'n} + \frac{1}{i\hbar} \int_0^t dt' e^{i\omega_{n'n} t'} H'_{n'n}, \tag{10}$$
where $\omega_{n'n} := (E_{n'}^{(0)} - E_n^{(0)})/\hbar$ and $H'_{n'n} := \langle n'^{(0)} | H' | n^{(0)} \rangle$.

- $3D$-Scattering: The wave function $\psi(r)$ of a particle scattering off a central potential $V(r)$ has the asymptotic form

$$\psi(\mathbf{r}) \to e^{ikz} + f(\theta)\frac{e^{ikr}}{r}, \quad r \to \infty, \tag{11}$$

where the incident wave is in the z-direction and $f(\theta)$ is called the scattering amplitude. The differential cross section is given by

$$d\sigma = |f(\theta)|^2 d\Omega. \tag{12}$$

and $d\sigma/d\Omega$ is called the differential cross section.

It can be shown that $f(\theta)$ can be expressed as

$$f(\theta) = \frac{1}{k}\sum_l (2l+1)P_l(\cos\theta)e^{i\delta_l}\sin\delta_l \tag{13}$$

where δ_l is the phase shift of the l-th partial wave

$$\psi(\mathbf{r}) = \frac{1}{2ik}\sum_l (2l+1)P_l(\cos\theta)\left[\frac{e^{ikr+2i\delta_l}}{r} - \frac{e^{-i(kr-l\pi)}}{r}\right] \tag{14}$$

As a result, the total cross section is given by

$$\sigma = \sum_l \sigma_l, \tag{15}$$

where

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l. \tag{16}$$

Examination Questions

Answer all questions.

You must show your work. No credits will be given if you don't show how you get your answers.

1. (25 points in total, 5 points each) Explain the following topics in QM:
   (a) Propagator
   (b) Bosons and Fermions
   (c) Fermi golden rules
   (d) Rabi oscillation
   (e) Spin-orbital coupling

   Answers should be at least 0.5 page each, including words, equations, and figures if the latter is necessary.

2. (12 points in total, 2 points each) Express each of the following quantities in terms of the fundamental constants $\hbar, e, c, m$ = electron mass, M = proton mass. Also give a rough estimate of numerical size for each.
   (a) Bohr radius (cm).
   (b) Binding energy of hydrogen
   (c) Compton wavelength of an electron (cm).
   (d) Classical electron radius (cm).
   (e) Electron rest energy (MeV).
   (f) Fine structure constant.

3. (10 points) Consider a perturbation $H' = \alpha x^4$ to the harmonic oscillator problem. Here $\alpha > 0$. Let m and ω be the mass and frequency of the oscillator.
   (a) (5 points) Show that the first-order correction to the unperturbed eigen-energies are

   $$E_n^1 = \frac{3\hbar^2\alpha}{4m^2\omega^2}(1 + 2n + 2n^2). \tag{17}$$

   Explain why no matter how small α is, the perturbation expansion will break down for some large enough n.

   (b) (5 points) If we make the perturbation time dependent, $H' = \alpha x^4 e^{-t^2/\tau^2}$ between $t = -\infty$ and $t = \infty$, what is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = \infty$?

   You might find the integral useful: $\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \sqrt{\pi/a}e^{b^2/4a}$.

4. (15 points, 5 points each) A hydrogen atom is placed under a constant external electric field along the z-axis:

   $$\mathbf{E} = E_0\mathbf{k}, \tag{18}$$

   The electron has charge $-e$, $e > 0$,and $|n, l, m\rangle$ denotes the eigenstates of the hydrogen atom without any perturbation. Consider the effects up to the first order in E_0.
   (a) Evaluate the commutator $[L_z, z]$, and use the result to explain the following selection rule: $\langle n', l', m'|z|n, l, m\rangle = 0$ if $m \neq m'$.

(b) Make use of the selection rule from part (a), and the fact the parity of the spherical harmonics $Y_{lm}(\theta, \phi)$ is $(-1)^l$ (you don't need to prove this), write down the perturbing Hamiltonian due to $\mathbf{E}$ as a matrix in the basis of $|n, l, m\rangle$ for $n = 2$. The matrix depends on only one parameter Δ, one of the non-zero matrix elements. Explain why and express the matrix in terms of Δ. (Note: Just write down the definition for Δ. You don't need to evaluate it.)

(c) What is the energy spectrum of the $n = 2$ states of this hydrogen atom, taking into account the first order energy corrections due the external electric field? Again you only need to use Δ to express the answers.

5. (20 points, 10 points each) A 3D nonrelativistic particle of mass m and energy E scatters quantum-mechanically in a central potential $V(r)$ given by

$$V(r) = \frac{\hbar^2}{2m}U(r), \quad U(r) = -2\left(\frac{\lambda}{\cosh \lambda r}\right)^2. \tag{19}$$

Given that the differential equation

$$\frac{d^2y}{dx^2} + k^2 y + \frac{2}{\cosh^2 x} y = 0 \tag{20}$$

has the solutions $y = e^{\pm ikx}(\tanh x \mp ik)$.

(a) Determine the wave function for the s-partial wave.

(b) Determine the s-wave contribution to the total cross section at energy E.

6. (18 points) A system of two particles each with spin 1/2 is described by an effectiveHamiltonian

$$H = A(s_{1z} + s_{2z}) + B\mathbf{s_1} \cdot \mathbf{s_2}, \tag{21}$$

where $\mathbf{s_1}, \mathbf{s_2}$ are the two spins, s_{1z}, s_{2z} are their z-components, and A, B are constants. Find all the eigenstates and energies of this Hamiltonian.

Qualifying Examination – Statistical Mechanics

Feb 22-23, 2025

Please explain the logic behind your answers.

Problem 1. *Ising model(15 points):* Considering the Ising model $H = -J\sum_{\langle i,j\rangle} \sigma_i\sigma_j$ where $\sigma_i = \pm 1$ and $\langle i,j\rangle$ denotes the nearest-neighbor pairs of sites.

1. For a system of coordinate number z, use mean-field approximation to find the critical temperature T_c below which spontaneous magnetization exists.
2. Show that the magnetic susceptibility $\chi \propto (T - T_c)^{-1}$ at $T \gg T_c$
3. Use entropy argument to show that in 1D there is no phase transition, i.e., $T_c = 0$.

Problem 2. *Free fermion problem(10 points):* Let the density of states of the electrons in a system be assumed to be a constant D for $\varepsilon > 0$ ($D = 0$ for $\varepsilon < 0$). Here, ε represents the energy. We assume the total number of electrons be equal to N. Here $N = D\int_0^\infty \frac{1}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon$.

1. Calculate the chemical potential μ at $T = 0K$.
2. When the system is in the regime of quantum degeneracy, estimate or derive the expression of the specific heat and show that it is proportional to T.

Problem 3. *Two-level system(15 points):* Consider a system of N distinguishable particles, which have two energy levels, $E_0 = -\mu B$ and $E_1 = \mu B$, for each particles. Here μ is the magnetic moment and B is the magnetic field. The particles populate the energy levels according to the classical distribution law.

1. Calculate the average energy of such system at temperature T
2. Calculate the specific heat of the system
3. Calculate the magnetic susceptibility

Problem 4. *Random walk(10 points):* A drunkard is kicked out of the pub. Being fully intoxicated, he walks forward and backward with equal probability. According to the senior high school math, the probability of finding him at position n after N step is $P(n,N) = C^N_{N_\rightarrow}\left(\frac{1}{2}\right)^{N_\rightarrow}\left(\frac{1}{2}\right)^{N_\leftarrow}$ where $N_\rightarrow$ and $N_\leftarrow$ represent the number of steps forward and backward, respectively. By definition, we have $N = N_\rightarrow + N_\leftarrow$ and $n = N_\rightarrow - N_\leftarrow$. Assume $N \gg n \gg 1$ so that the Stirling formula can be used to approximate all large factorials: $\lim_{N\gg 1} N! \approx N\ln N - N$. Show that $P(n,N)$ can be reduced to the Gaussian distribution : $P(n,N) \sim \frac{1}{\sqrt{N}} exp\left[-\frac{n^2}{4N}\right]$.

Problem 5. *Spins and vacancies on a surface and the negative temperature(20 points):* Consider a surface with N sites and a collection of non-interacting spin $\frac{1}{2}$ particles. For each site the energy $\varepsilon = 0$ if there is a vacancy and $\varepsilon = -W$ if there is a particle present, where $-W < 0$ is the binding energy.

1. Let $N_{\pm}$ be the number of particle with spin $\pm\frac{1}{2}$, $Q = N_+ + N_-$ be the number of spins, N_0 be the number of vacancies, $M = N_+ - N_-$ be the surface magnetization. We have $N_+ + N_- + N_0 = N = Q + N_0$. In the microcanonical ensemble, compute the entropy $S(Q, M)$. (Hint: Calculate the number of states available to the system and express N_+, N_-, N_0 with Q and M.

2. Let $q = \frac{Q}{N}$ be the dimensionless particle density and $m = \frac{M}{N}$ be the dimensionless magnetization density. Assuming that we are at the thermodynamic limit where N, Q and M all tend to infinity, but with q and m finite. Find the temperature $T(q, m)$. (Hint: Use Stirling's formula $\ln N! \approx N \ln N - N$.)

3. Show explicitly that T can be negative for this system. What does negative T mean? What physical degrees of freedom have been left out that would avoid this strange property?

Problem 6. *Bosons in Harmonic traps(30 points):* Let us consider a particle in the anisotropic harmonic-oscillator potential $V(\boldsymbol{r}) = \frac{1}{2}\left(K_x x^2 + K_y y^2 + K_z z^2\right)$. Therefore, we can consider the system as three independent harmonic oscillators in three different directions x, y and z. The corresponding energy levels can be parameterized by three non negative integers (n_x, n_y, n_z) as

$$E(n_x, n_y, n_z) = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z. \tag{1}$$

1. Let us consider n_i are continuous variables and neglect the zero-point energies (The $\frac{1}{2}\hbar\omega_i$ in $E(n_x, n_y, n_z)$.), we can simplify the energy as $E \approx \varepsilon_x + \varepsilon_y + \varepsilon_z$ where $\varepsilon_i = n_i\hbar\omega_i$. If we define $G(E)$ as the number of states below energy a specific energy E. Derive the expression of $G(E)$ in terms of E, $\omega_x, \omega_y, \omega_z$ and $\hbar$. (Hint: There are several way to derive this result. One approach is to consider the problem in the (n_x, n_y, n_z) space. What is the geometric meaning of the states below energy E?)

2. Another way to understand density of state is $g(E) = \frac{dG(E)}{dE}$. Use the above expression to get the density of states.

3. We can evaluate the particles in the excited states using

$$N_{ex} = \int_0^\infty dE g(E) \frac{1}{e^{E/k_BT} - 1}. \tag{2}$$

   Here, we assume N_{ex} reaches its maximum for $\mu = 0$. The definition of the transition temperature, T_c, of Bose-Einstein condensation is when the total number of particles can be just accommodated in excited states. That is,

$$N = N_{ex}(T = T_c, \mu = 0) = \int_0^\infty dE g(E) \frac{1}{e^{E/k_BT} - 1}. \tag{3}$$

   $\int_0^\infty dx \frac{x^{\alpha-1}}{e^x - 1} = \Gamma(\alpha)\zeta(\alpha)$. Here, $\zeta(\alpha) = \sum_{n=1}^\infty n^{-\alpha}$ is the Riemann zeta function and $\Gamma(\alpha)$ is the gamma function. Derive the transition temperature of this system and express the result using $N, \omega_x, \omega_y, \omega_z$ and the Riemann zeta function.