

Electrodynamics Qualifying Exams

September 2023

You must provide the details or reasonings to justify your answers.

Problem 1: Simple Questions (5% each = 35% total)

1. Write down the four Maxwell's equations. Specify all the numerical values of the relevant physical constants and the units you adopt.
2. Why are electromagnetic waves transverse in the vacuum?
3. In one particular inertial frame, the electric and magnetic fields are measured as $\vec{E} = (9, -1, 7)$ and $c\vec{B} = (5, 0, -2)$ in some units, respectively. For the same field configuration can you measure in a different frame $\vec{E}' = (2, 0, 4)$ and $c\vec{B}' = (3, 5, 2)$?
4. Explain the terms Lorentz gauge and Coulomb gauge.
5. Explain the term Green's function.
6. Explain the terms Dirichlet and Neumann boundary conditions.
7. How many real degrees of freedom has a dipole moment? Explain your answer.

Problem 2: Simple derivations (10% each = 20% total)

1. Derive the conservation of charge from the Maxwell equations.
2. Derive the wave equation for electromagnetic fields from the Maxwell equations.

Problem 3: Moving charge (15%)

What are a point charge's electric and magnetic fields at a constant velocity \vec{v} ?

Problem 4: Image Charges (18%)

Consider as volume the whole three-dimensional space. Centered at the origin is a grounded metal sphere with radius R . At $\vec{x} = (0, 0, d)$ we position a point charge with charge q . On the surface of the metal sphere and at $\vec{x} \rightarrow \infty$ the electric potential vanishes.

1. (5%) Find the electric potential outside the sphere and check that it fulfills the boundary conditions.
2. (8%) Calculate the surface charge density

$$\sigma = -\epsilon_0 \hat{n} \cdot \vec{\nabla} \phi, \quad (1)$$

with \hat{n} the normal vector on the surface S .

3. (5%) Calculate the total charge

$$Q = \int \sigma \, dS \quad (2)$$

on the sphere. You can use Gauss' theorem to calculate Q .

Problem 5: Multipole Moments (12%)

Consider two homogeneously charged, solid hemispheres with radius R which are separated in the x - y -plane by a negligible slit. The upper hemisphere shall have total charge $+Q$ and the lower hemisphere shall have total charge $-Q$.

1. (3%) Write down an expression for the charge density, ρ , of the two hemispheres.
2. (4%) Calculate either the spherical or the Cartesian monopole moment.
3. (5%) Calculate either the spherical or the Cartesian dipole moment.

Quantum Mechanics Fall 2023

Qualifying Exam

You must show your work. No credits will be given if you don't show how you get your answers.

Boldface characters like \mathbf{v} refer to vectors ($= \vec{v}$).

You may use the following formula:

- The Schrödinger equation:

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = \hat{H} |\psi(t)\rangle$$

where \hat{H} is the Hamiltonian $\frac{\hat{\mathbf{p}}^2}{2\mu} + V(\hat{\mathbf{r}})$ and μ is the mass of the particle.

For energy eigenstates, this reduces to the time-independent Schrödinger equation (in spherical coordinates)

$$-\frac{\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \psi(\mathbf{r}) + \langle \mathbf{r} | \frac{\hat{\mathbf{L}}^2}{2\mu r^2} | \psi \rangle + V(\mathbf{r}) \psi(\mathbf{r}) = E \psi(\mathbf{r})$$

$\hat{\mathbf{L}}$ is the orbital angular momentum operator, $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$.

- For a 1-D simple harmonic oscillator (SHO), $\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$:

The raising and lowering operators are

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p}_x \right), \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p}_x \right)$$

and $[\hat{a}, \hat{a}^\dagger] = 1$. The operators get their names from the facts that

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle,$$

where $|n\rangle$'s are the energy eigenstates of the 1D SHO.

- The Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Time-independent Perturbation: Consider $\hat{H} = \hat{H}_0 + \hat{H}_1$, where \hat{H}_1 is a small perturbation. The first order energy correction to the n-th energy eigenvalue of \hat{H}_0 is

$$E_n^{(1)} = \langle n^{(0)} | \hat{H}_1 | n^{(0)} \rangle$$

where $|n^{(0)}\rangle$ is the n-th eigenstate of \hat{H}_0 .

The second order energy correction is

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle n^{(0)} | \hat{H}_1 | m^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}}.$$

- Time-dependent Perturbation: Consider $\hat{H} = \hat{H}_0 + \hat{H}_1(t)$, where $\hat{H}_1(t)$ is a small perturbation. The state $|\psi(t)\rangle$ can be expanded in terms of the unperturbed eigenstates $|n^{(0)}\rangle$:

$$|\psi(t)\rangle = \sum_{n=0} d_n(t) e^{-i \frac{E_n^{(0)} t}{\hbar}} |n^{(0)}\rangle.$$

If initially ($t = 0$) the particle is at state $|i^{(0)}\rangle$, the transition amplitude for the particle to be in state $|f^{(0)}\rangle$ at time t is

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar} \int_0^t dt' \langle f^{(0)} | \hat{H}_1(t') | i^{(0)} \rangle e^{i \frac{(E_f^{(0)} - E_i^{(0)}) t'}{\hbar}}.$$

- 3-D Scattering: The wave function $\psi(\mathbf{r})$ of a particle scattering off a potential $V(\mathbf{r})$ has the asymptotic behavior

$$\psi(\mathbf{r}) = A e^{ikz} + A f(\theta, \phi) \frac{e^{ikr}}{r}, \quad r \rightarrow \infty,$$

where the incident wave is in the z-direction.

- Born Approximation: Useful for high-energy scattering.

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int d^3\mathbf{r}' V(\mathbf{r}') e^{-i\mathbf{q}\cdot\mathbf{r}'},$$

where m is the mass of the scattering particle, and $\hbar\mathbf{q} = \hbar\mathbf{k}_f - \hbar\mathbf{k}_i$ is the momentum transferred from the initial state $e^{i\mathbf{k}_i\cdot\mathbf{r}}$ to the final state $e^{i\mathbf{k}_f\cdot\mathbf{r}}$.

- Partial wave expansion: Useful for low-energy scattering. For spherically-symmetric $V(r)$,

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) \frac{e^{i\delta_l}}{k} \sin \delta_l P_l(\cos \theta),$$

where δ_l is the phase shift of the l -th partial wave, and $P_l(\cos\theta)$ is the Legendre polynomial, which relates to the spherical harmonics $Y_{l,m}(\theta, \phi)$ as

$$Y_{l,0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta).$$

- Useful Gaussian integrals:

$$\int_{-\infty}^{\infty} \exp(-ax^2 + bx) dx = \exp\left(\frac{b^2}{4a}\right) \sqrt{\frac{\pi}{a}}$$

1. Spin-1/2 systems: (5 points each)

- (a) **Explicitly derive** (not just plug in the answer) the eigenvector $|+\mathbf{n}\rangle$ of the spin momentum operator $\hat{S}_n = \hat{\mathbf{S}} \cdot \hat{\mathbf{n}}$ along a unit vector $\hat{\mathbf{n}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. Show that (in the S_z -basis)

$$|+\mathbf{n}\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

up to an overall phase.

[*Hint:* You can use the half-angle formula

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}, \quad \cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}.$$

]

- (b) Consider the spin singlet state formed by two spin-1/2 particles. Alice and Bob measure the spin of each particle along the unit vectors \mathbf{a} and \mathbf{b} , respectively. Let $S_a^{(1)}$ is the spin of Particle 1 along \mathbf{a} , and $S_b^{(2)}$ is the spin of Particle 2 along \mathbf{b} . Use the result from part (a), show that the expectation value of the product of Alice and Bob's measurements

$$\langle S_a^{(1)} S_b^{(2)} \rangle = -\frac{\hbar^2}{4} \cos \theta_{ab},$$

where θ_{ab} is the angle between \mathbf{a} and \mathbf{b} .

- (c) At time $t = 0$, an electron and a positron are in a state with total spin = 0. The electron-positron pair are placed in a uniform magnetic field along the z -axis, $\mathbf{B} = B_0 \hat{k}$. Find the state as a function of time. Show that the total spin of the system oscillates between spin-0 and spin-1, and find the period of oscillation.

[*Note:* Express your answer in terms of the electron charge $-e$ ($e > 0$), mass m_e , g-factor g , and other constants in this problem. The magnetic moment $\boldsymbol{\mu}$ of a charged particle of spin \mathbf{S} , mass m , charge q , and g-factor g is $\boldsymbol{\mu} = \frac{gq}{2mc} \mathbf{S}$.]

2. 1D Simple Harmonic Oscillator (SHO): Consider a 1D SHO with mass m and frequency ω .

- (a) A coherent state $|\alpha\rangle$ is defined as the eigenstate of the lowering operator $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Do α 's have to be real? Explain why. (3 points)

- (b) Show that $\Delta x \Delta p_x$ for a coherent state saturates the lowest limit allowed by the Uncertainty Principle. (8 points)
- (c) What is the expectation value of energy for a coherent state $|\alpha\rangle$? (4 points)
- (d) Use the information from part (a) and (b), derive $\Delta x \Delta p_x$ for the ground state of a 1D SHO. (3 points)

3. SHO and perturbation:

- (a) Consider a 1D SHO with mass m and frequency ω . Suppose a perturbation of the form $H_1 = b\hat{x}^4$ ($b > 0$) is added to the 1D SHO. Find out the first order energy correction to the n -th energy level of the SHO. What is the condition for this first order result to be a valid energy correction? (5 points)
- (b) From part (a), explain why the first order energy correction must break down when n is large enough, no matter how small b is. (2 points)
- (c) If we make the perturbation time-dependent, $H_1 = b\hat{x}^4 \exp\left(-\frac{t^2}{\tau^2}\right)$ ($\tau > 0$), starting from $t = -\infty$. The SHO was originally at the ground state at $t = -\infty$. What is the probability for the SHO to transit to an excited state $|n\rangle$ at $t = +\infty$? (5 points)

4. A particle of mass m is placed under a 3D potential $V(\mathbf{r})$ which is a central potential that vanishes as $r \rightarrow \infty$. The wave function of the particle is of the form

$$\psi_E(\mathbf{r}) = A \exp\left(-\frac{r}{a_0}\right),$$

where $a_0 > 0$ is a constant. (5 points each)

- (a) What is the expectation value of angular momentum of the particle?
- (b) If ψ_E is an energy eigenstate, find out the energy eigenvalue E .
- (c) Having found E , what is $V(r)$ at finite r .

5. Angular momentum:

- (a) Consider a particle described by a wave function localized at $\mathbf{r} = (x_0, y_0, z_0)$, and its total angular momentum is $\mathbf{J} = (J_x, J_y, J_z)$. Define an operator $\hat{O} = \exp\left(i\frac{J_z\phi}{\hbar}\right)x^2 \exp\left(-i\frac{J_z\phi}{\hbar}\right)$, write down the expectation value $\langle O \rangle$ for the state of the particle in terms of x_0, y_0 and ϕ . (5 points)

- (b) What is the spin-orbit interaction? Please briefly explain the origin, and show that for each energy level of a hydrogen atom,

$$E_{\text{SO}} \sim \alpha^2 E^{(0)},$$

where E_{SO} is the additional energy from the spin-orbit interaction, $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant, and $E^{(0)}$ is the energy eigenvalue from the Coulomb potential. (10 points)

[*Hint:* The magnetic field produced by a charge q with velocity \mathbf{v} is $\mathbf{B}(\mathbf{r}) = \frac{q}{c} \frac{\mathbf{v} \times \mathbf{r}}{r^3}$, and $E^{(0)} \sim \alpha^2 (m_e c^2)$ for hydrogen atoms, where m_e is the mass of the electron.]

- (c) Consider an electron in a hydrogen atom in the state $Y_1^{+1} |+\rangle$, express the result of $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} Y_1^{+1} |+\rangle$ in terms of Y_l^m and $(|+\rangle, |-\rangle)$. (5 points)
6. 3D Scattering: Consider a particle of mass m moving in 3D scattering against a potential $V(\mathbf{r}) = V_0 \exp(-r^2/R^2)$, where V_0 and $R > 0$ are constants.
- (a) Use the Born approximation, calculate the the scattering amplitude $f(\theta)$ for the potential $V(\mathbf{r}) = V_0 \exp(-r^2/R^2)$. (10 points)
[*Note:* You should express your answer in terms of the initial $k(=|\mathbf{k}_i|)$ and θ .]
- (b) Derive the s-wave scattering length in the limit of zero energy for the potential $V(\mathbf{r}) = V_0 \exp(-r^2/R^2)$. (10 points)

Qualification Exam. Problem Set
Classical Mechanics

Fall, 2023

1. For a one-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2}$$

- (a) (10 points) Show that there is a constant of the motion

$$D = \frac{pq}{2} - Ht$$

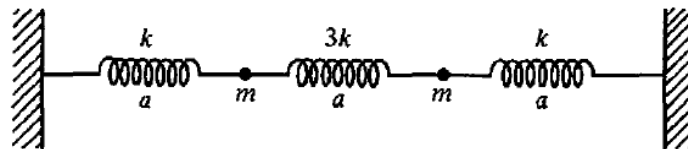
- (b) (10 points) As a generalization of part (a), for motion in a plane with the Hamiltonian

$$H = |\mathbf{p}|^n - ar^n$$

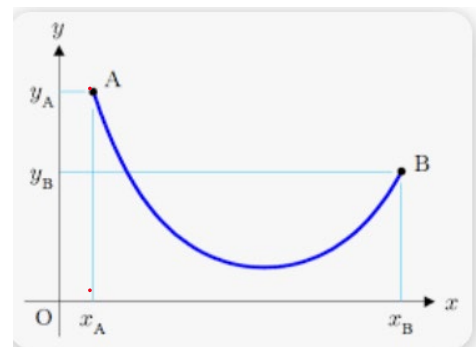
where \mathbf{p} is the vector of the momenta conjugate to the Cartesian coordinates, show that there is a constant of the motion

$$D = \frac{\mathbf{p} \cdot \mathbf{r}}{n} - Ht.$$

2. (20 points) Two particles of mass m move in one dimension at the junction of three springs, as shown in the figure. The springs all have unstretched lengths equal to a , and the force constants are $k, 3k, k$, respectively. Find the eigenfrequencies and normal modes of the system.



3. (20 points) A rope of indefinite length passes freely over pulleys at heights y_A and y_B above the plane surface of earth, with a horizontal distance $x_B - x_A$ between them. Find the curve between the pulleys assuming the rope has a uniform linear mass density.



4. A particle of mass m described by one generalized coordinate q moves under the influence of a potential $V(q)$ and a damping force proportional to its velocity as $-2m\gamma\dot{q}$.

- (a) (5 points) Show that the following Lagrangian $L = e^{2\gamma t} \left(\frac{1}{2} m \dot{q}^2 - V(q) \right)$

gives the desired equation of motion.

(b) · (5 points) Obtain the Hamiltonian $H(q, p, t)$ for this system.

(c) · (5 points) Consider the following generating function

$$F = e^{\gamma t} qP - QP$$

Obtain the canonical transformation from (q, p) to (Q, P) and the transformed Hamiltonian $K(Q, P, t)$.

(d) · (5 points) Pick $V(q) = \frac{1}{2} m\omega^2 q^2$ as a harmonic potential with a natural frequency ω . Show that the transformed Hamiltonian yields a constant of motion.

5. A particle of mass m in a Kepler central potential $V(r) = -k/r$ has orbits

described by $\frac{1}{r} = \frac{mk}{l^2} (1 + \epsilon \cos\theta)$, where $\epsilon = \sqrt{1 + \frac{2l^2 E}{mk^2}}$

(a) · (10 points) Suppose the particle is initially in a parabolic orbit. An impulse is applied at periastron ($\theta = 0$) to place the particle in a circular orbit. Give the energy and angular momentum of the circular orbit in terms of the energy and angular momentum of the initial parabolic orbit, and characterize completely the required impulse. Draw an effective-potential diagram that shows the transition between the two orbits.

(b) · (10 points) Suppose the particle is initially in an arbitrary elliptical orbit. An impulse is applied at $\theta = \pi/2$ to place the particle in a circular orbit. Give the energy and angular momentum of the circular orbit in terms of the energy and angular momentum of the initial orbit, and characterize completely the required impulse. Draw an effective potential diagram that shows the transition between the two orbits.

Qualifying Examination – Statistical Mechanics

Sep 16-17, 2023

Please explain the logic behind your answers.

Problem 1. *Freely Jointed Chain (Gaussian model) and its mean field picture (40 points)*

Polymers are large (high molar mass) molecules composed of a large number of monomers bonded together to form a chain. In reality, the monomers are bonded covalently. That is, the potential energy between two monomers should be a complex function of the bonding angle between two nearby monomers. Let's make some assumptions to simplify our problem which leads to the freely jointed model. (We might over simplify the problem but let's just try it first to get a feeling about what kind of problem we are facing. Then we can ask how to put back more ingredients to make our analysis more realistic.)

- We assume the joint of the monomers can rotate freely, *i.e.* the potential energy is independent of the bonding angle, ϕ , between two nearby monomers.
- We assume the monomers can overlap with each other in space and have no interaction between each other.

The model with above two assumptions is the freely jointed chain model. It is the idealization of polymers analogous to ideal gas model for gases.

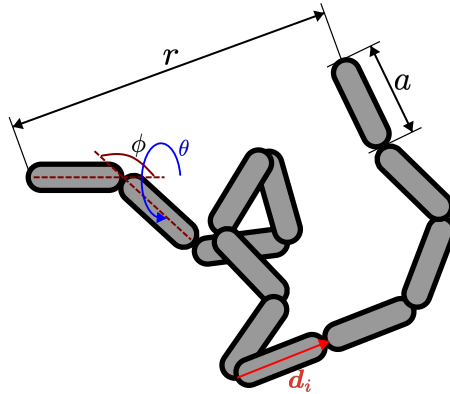


Figure 1: Schematic picture of a single chain polymer molecular formed by the monomers represented by the grey oval.

A key property that we are interested in is the size of the polymer. Usually the polymer coils up. The way they coils up will be a competition between the entropy and the energetics. Therefore, the size of the polymer is not simply the number of monomers, N , times the size of the monomer, a . Instead, we need to use some statistical description to characterize the size of the polymer. Usually,

we use the root-mean-square end-to-end distance, $\langle r^2 \rangle^{1/2}$, as a measure for the size of the polymer. Here, r is the distance from one end of the polymer to the other in three-dimensional space. The average is taken over all possible ways the polymer coils. One of the configuration is simple, if all the monomers are aligned in one direction, the end-to-end distance $r_{straight} = Na$. However, this is just one of the possible value of r , once the polymer coils, $\langle r^2 \rangle^{1/2} < Na$.

1. It is interesting to observe that the freely jointed model in one-dimension is actually the simple one-dimensional random walk we discussed during the lecture. The step size is just the size of the monomer. The one-dimensional constraint restricts $\phi = 0/\pi$. Let's start in this simple limit and use x to represent the end-to-end distance in one-dimensional case. Derive the probability distribution $P_{1D}(x, N)$ when the number of monomers N is large.
2. For the three-dimensional case, we assume the orientation is completely random and the vector from monomer i and the vector for monomer $i + 1$ are uncorrelated. *i.e.* $\langle \mathbf{d}_i \cdot \mathbf{d}_j \rangle_{i \neq j} = 0$. Therefore, we expect N to be distributed evenly $N_x = N_y = N_z = \frac{N}{3}$. Here, N_x is a rough definition of the monomer belongs to the monomer in x direction. We can simply consider the projection of the monomer to the x, y, z direction. If the projection to x direction is has the largest size, we said it is a x monomer that should be counted in N_x . Derive $\langle r^2 \rangle^{1/2}$ and express it using N and a . This is a simple estimation for the size of the polymer. (Sometimes you will see people use the radius of gyration r_g^2 to estimate the size of the polymer. r_g is the average distance between monomers and the center of mass. It turns out the length scale, $\langle r_g^2 \rangle$, will be proportional with $\langle r^2 \rangle$. We will not discuss the calculation here, but just mention the fact that using the root mean square of the end-to-end distance capture the essential information for the size of the polymer.)
3. Now we have a rough idea about the freely jointed model. Let's assume we are in three-dimensional world. In the last problem, we should be able to have the probability distribution function $P_{3D}(r, N)$. Using the microcanonical ensemble, we should expect the probability distribution to related to the phase space volume as

$$P_{3D}(r, N) = \frac{\Omega(r, N)}{\int dr \Omega(r, N)} \equiv \frac{\Omega(r, N)}{A_N}. \quad (1)$$

Here, $\Omega(r, N)$ is the phase space volume for the size N polymer with the end-to-end distance r . A_N is the integral of $\Omega(r, N)$ over r . Use (1) to find the expression of the entropy, $S(r, N)$, for the size N polymer with the end-to-end distance r using N, a, r, k_B and A_N .

4. With the expression of entropy, we can derive the Helmholtz free energy of the system using $F(r, N) = U(r, N) - TS(r, N)$. Here, we are considering the freely jointed model, so there is no internal energy. The Helmholtz free energy is just $F(r, N) = -TS(r, N)$. Here is the interesting part, the free energy depends on r just like the spring with elastic energy ($E \propto k(\delta x)^2$). We can have the elasticity constant of the polymer. Find the expression of the elasticity constant k for the polymer described by the freely jointed model.
5. A rubber is an composite object with entangled polymers. They are difficult to analyze during the qualify exam. However, from the simple calculation we achieved here we can explain some interesting physics. Consider a piece of rubber hanging a massive object below it. Suppose the rubber is strong enough such that the massive object just hang in the air. Next, if we heat up the rubber (hopefully we did not melt it.), what would you expect? Will the massive object be lifted higher? Or the other way around?

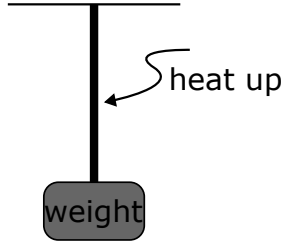


Figure 2: Schematic plot for the massive object hanging in the air.

6. Till now, we assume the non-interacting limit. So the free energy only consider the entropic effect. Now, we would like to extend the understanding to the case that the monomers can actually interact with each other. In order to proceed, we make the following assumptions:

- In Fig. 1, we can clearly see the monomer is a rod like object which does not have rotational symmetry (it is not a sphere). However, **let's put that aside and assume the monomers are spherically symmetric objects with volume v that forbidden the overlap of two monomers.**
- We can define the packing ratio, μ , of the monomers for the polymer as

$$\mu = \frac{Nv}{r^3}. \quad (2)$$

We drop the irrelevant coefficients just to get an estimation about the physics. It is the ratio of the total volume of the monomers and the volume given by the "size" of the polymer.

Since the interaction avoids two monomers to overlap in space. We can assume that the interaction between the monomers are pair-wise interaction. (They are just one monomer interacts with another one to satisfy the no-overlapping condition.) Estimate the interaction energy, $U_{int}(r, N, T)$, at temperature T for a polymer with length N at large N . (Hint: Think about the meaning of the packing ratio. How to estimate the pair-wise interactions using the packing ratio?)

7. Before we do any calculation, let's think about the effect of each term. The entropic contribution favors the value of small r to minimize the improved free energy. On the other hand, since the monomers repel each other due to the interaction energy, the $U_{int}(r, N, T)$ term prefers the value of large r in order to minimize the $F_{improved}(r, N, T)$. Therefore, the two effects competes with each other and fix the characteristic size of the polymer with length scale r^* . Use the above result to improve the free energy estimation by including the interaction between the monomers. We should have $F_{improved}(r, N, T) = U_{int}(r, N, T) + F_{entropic}(r, N, T)$. Find r^* as a function of v, a, N .
8. $r_{ideal} \sim N^{\frac{1}{2}}$, $r^* \sim N^\alpha$. Compare r_{ideal} and r^* at large N limit. Is $\alpha > \frac{1}{2}$, $\alpha < \frac{1}{2}$ or $\alpha = \frac{1}{2}$? Why is it a reasonable result?

Problem 2. *A simplified model of hemoglobin(10 points):* Hemoglobin is a protein in the blood that carries oxygen from the lungs to the muscles. It is formed by four units. Each unit can carry an O_2 molecule or not. For a hemoglobin, there will be 2^4 configurations as shown in Fig.3. (Each unit has two choices, carries nothing or an O_2 molecule. Therefore we have 2^4 possible choices.) As described in the caption of Fig.3, we can model the hemoglobin by assigning the binding energy $-\varepsilon_0$ and the additional gain, $-J$, if two nearby unit both bind with O_2 (We try to pack O_2 to our hemoglobin as much as possible, a simple minded physicist guess). For example, considering the

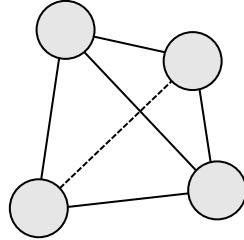


Figure 3: Schematic model of hemoglobin. The gray circles represent the unit that can carry an O_2 molecule. We assume the four units form a tetrahedral network structure. We can approximate the hemoglobin using a simplified model: The energy gain for an O_2 to bind with a unit is $-\varepsilon_0$. If two nearby units both contain O_2 , the energy can further reduced by energy $-J$.

configuration in Fig. 4, the total energy of the simplified model is $(-\varepsilon_0) \times 2 + (-J) \times 1$. Consider

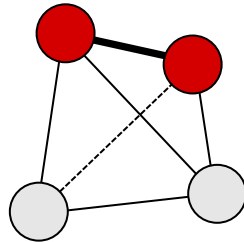


Figure 4: One particular configuration of hemoglobin. The red circles represent the units that are occupied by O_2 molecules. The thick line represent the additional energy reduction, $-J$, for nearby units are occupied by O_2 molecules. There will be other 5 configurations with equal energy.

we have $N/4$ hemoglobin, labeled by $\alpha = 1 \sim \frac{N}{4}$. On each hemoglobin, we have 4 units which can bind with O_2 molecules. We can use $\tau_{\alpha,i} = 0, 1$ for $i = 1 \sim 4$ to denote whether the i -th unit on the α -th hemoglobin is occupied by an O_2 molecule or not.

1. Write down the expression of the grand canonical partition function of the system with $\frac{N}{4}$ hemoglobin. Express the result using $x = \exp\left[\frac{\varepsilon_0 + \mu}{k_B T}\right]$ and $y = \exp\left[\frac{J}{k_B T}\right]$. (Hint: we consider the molecules of hemoglobin to be independent. So, we can factorize the partition function of the system into products of partition functions of individual hemoglobin molecule. Then, within each Hemoglobin, we can enumerate all the possible configurations and the corresponding energy to construct our partition function. Be careful about the number of configurations with equal energy. The degeneracy of energy in this system is not very regular, so one needs to specify them explicitly.)
2. Evaluate the average number $\langle M \rangle = \sum_{\alpha=1}^{N/4} \sum_{i=1}^4 \langle \tau_{\alpha,i} \rangle$ of adsorbed molecules as a function of x, y .

Problem 3. Free fermion problem(10 points): Let the density of states of the electrons in a system be assumed to be a constant D for $\varepsilon > 0$ ($D = 0$ for $\varepsilon < 0$). Here, ε represents the energy. We assume the total number of electrons be equal to N . Here $N = D \int_0^\infty \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} d\varepsilon$.

1. Calculate the chemical potential μ at $T = 0K$.
2. When the system is in the regime of quantum degeneracy, estimate or derive the expression of the specific heat and show that it is proportional to T .

Problem 4. Two-level system(10 points): Consider a system of N distinguishable particles, which have two energy levels, $E_0 = -\mu B$ and $E_1 = \mu B$, for each particles. Here μ is the magnetic moment and B is the magnetic field. The particles populate the energy levels according to the classical distribution law.

1. Calculate the average energy of such system at temperature T
2. Calculate the specific heat of the system

Problem 6. Bosons in Harmonic traps(30 points): Let us consider a particle in the anisotropic harmonic-oscillator potential $V(\mathbf{r}) = \frac{1}{2} (K_x x^2 + K_y y^2 + K_z z^2)$. Therefore, we can consider the system as three independent harmonic oscillators in three different directions x, y and z . The corresponding energy levels can be parameterized by three non negative integers (n_x, n_y, n_z) as

$$E(n_x, n_y, n_z) = \left(n_x + \frac{1}{2}\right) \hbar\omega_x + \left(n_y + \frac{1}{2}\right) \hbar\omega_y + \left(n_z + \frac{1}{2}\right) \hbar\omega_z. \quad (3)$$

1. Let us consider n_i are continuous variables and neglect the zero-point energies (The $\frac{1}{2}\hbar\omega_i$ in $E(n_x, n_y, n_z)$), we can simplify the energy as $E \approx \varepsilon_x + \varepsilon_y + \varepsilon_z$ where $\varepsilon_i = n_i \hbar\omega_i$. If we define $G(E)$ as the number of states below energy a specific energy E . Derive the expression of $G(E)$ in terms of $E, \omega_x, \omega_y, \omega_z$ and \hbar . (Hint: There are several way to derive this result. One approach is to consider the problem in the (n_x, n_y, n_z) space. What is the geometric meaning of the states below energy E ?)
2. Another way to understand density of state is $g(E) = \frac{dG(E)}{dE}$. Use the above expression to get the density of states.
3. We can evaluate the particles in the excited states using

$$N_{ex} = \int_0^\infty dE g(E) \frac{1}{e^{E/k_B T} - 1}. \quad (4)$$

Here, we assume N_{ex} reaches its maximum for $\mu = 0$. The definition of the transition temperature, T_c , of Bose-Einstein condensation is when the total number of particles can be just accommodated in excited states. That is,

$$N = N_{ex}(T = T_c, \mu = 0) = \int_0^\infty dE g(E) \frac{1}{e^{E/k_B T} - 1}. \quad (5)$$

$\int_0^\infty dx \frac{x^{\alpha-1}}{e^x - 1} = \Gamma(\alpha)\zeta(\alpha)$. Here, $\zeta(\alpha) = \sum_{n=1}^\infty n^{-\alpha}$ is the Riemann zeta function and $\Gamma(\alpha)$ is the gamma function. Derive the transition temperature of this system and express the result using $N, \omega_x, \omega_y, \omega_z$ and the Riemann zeta function.