Electrodynamics Qualifying Examination, Sep., 2022.

Specify the units you use (the SI units are preferred).
You must provide the details or reasonings to justify your answers.

(5% each)

1. Write down the four Maxwell's equations. Specify all the numerical values of the relevant physical constants and the units you adopt.

2. On a sunny day in the dry Sahara desert, how do you determine the direction of your polarizer?

3. How do you classify magnetic substances? Name ONE material for each category.

4. Why are the EM waves transverse in the vacuum?

5. Explain what phase- and group-velocities are, and provide one example where the two are different.

6. What are perfect conductors and superconductors, and how do they differ?

7. In one particular inertial frame, the electric and magnetic fields are measured as $\vec{E} = (9, -1, 7)$ and $c\vec{B} = (5, 0, -2)$ in some units, respectively. In a different frame, it is known that $\vec{E}' = (2, 0, 4)$. Give one possible $c\vec{B}'$.

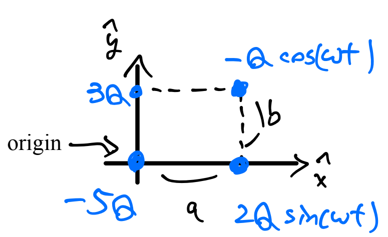


Fig-1

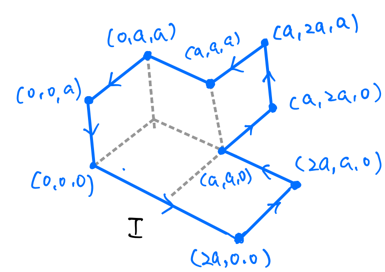


Fig-2

1. (8+7%)
Four time-dependent point charges are located at different positions, as shown in figure-1. (a) What is the electric field observed at location $(L, 0, -L)$? (assuming $L \gg a, b$) (b) What is the total time-averaged radiation power of this system?

2. (5+5%)
(a) Calculate the magnetic dipole moment from the electric current loop shown in figure-2. (b) What is the magnetic field far away from the origin?

3. (10%)
What are a point charge's electric and magnetic fields at a constant velocity $\vec{v}$?

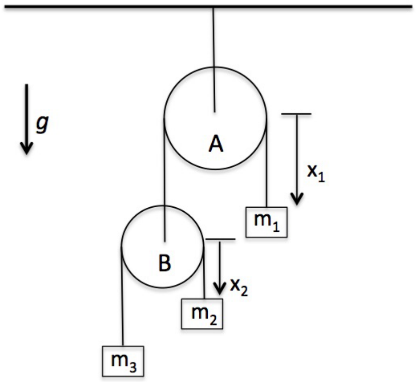
4. (8+7%)
A rectangular waveguide has a cross-section of $a \times a$, filled with a dielectric substance of permeability μ and permittivity ϵ. (a) What is the range of frequencies where only TE_{10} and TE_{01} modes can be transmitted in the waveguide? (b) What are the possible issues if the online course signals are broadcast using a rectangular waveguide?

5. (5+5+5%)
A long solid dielectric cylinder of radius a and length L, $L \gg a$, carries a uniform charge density ρ, and it rotates about its axis with an angular velocity $\omega(\omega a \ll c)$. (a) What are the electric and magnetic fields outside the cylinder near the midpoint of the cylinder? (b) What is the vector potential inside and outside near the midpoint of the cylinder? (c) What is the total angular momentum carried by the EM fields?

Qualification Exam. Problem Set
Classical Mechanics

Fall, 2022

1. A compound Atwood machine is composed of three masses m_1, m_2, and m_3 attached to two massless ropes through two massless pulleys A and B of radii r_A and r_B, respectively. Pulley A is attached to a stationary ceiling. The lengths l_a and l_b of the ropes around pulleys A and B are fixed, and the ropes do not slip as the pulleys rotate. The system is in a constant gravitational field with acceleration g.



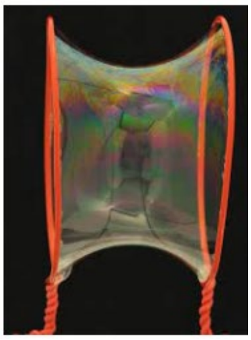
Take as two generalized coordinates x_1 and x_2 the distance from the center of pulley A to mass $m_1$and the distance from the center of pulley B to mass m_2, respectively as shown if the figure..

(a)、 (10 points) Write down the Lagrangian L of the system.

(b)、 (5 points) Derive the equations of motion for x_1 and x_2. You do not need to solve them explicitly.

(c)、(5 points) Next assume that pulley A is a uniform disk with mass $M$and radius r_A. Write down the Lagrangian L for this system.

2. A soap film is built between two circular wires of radius R that are separated by a distance L. Air is allowed to flow through the wires. Please



(a)、(10 points) find the shape of film.

(b)、 (10 points) show that there exists a upper bound for L beyond which the film becomes unstable.

3. (20 points)Derive Kepler's third law: $\tau^2 \propto a^3$ where τ is the period and a the semi-major axis (長軸半長) by the following steps:

(a)、draw an elliptic orbit with semi-major and minor axes a, b and put the sun on one of its focus (焦點),

(b)、 denote the speed of planet at perihelion (近日點) and aphelion (遠日點) by v_p and v_a,

(c)、 write down the conservations of angular momentum ℓ and mechanical energy at perihelion and aphelion,

(d)、 given that the area of ellipse equals $ab\pi$, relate τ to ℓ.

4. A particle of mass m moves along the x-axis in a potential $U(x) = -ax^2 + \lambda x^4$, where a and λ are positive constants.

(a)、(6 points) Write down the Hamiltonian $H(p, x)$ describing this system and find expressions for the time derivatives $\dot{x}$ and $\dot{p}$ using Hamilton's equations of motion.

(b)、 (7 points) Find the equilibrium values of x, and for each one determine if it is stable or unstable. Find the frequency of small oscillations ω around the stable points.

(c)、 (7 points) State Hamilton's principle of least action. Consider a trajectory where the particle sits stationary at a stable equilibrium point for a total time T. What is the classical action S along this trajectory?

5. (20 points)The Hamiltonian of a (pseudo) relativistic one-dimensional oscillator is given by

$$H = \sqrt{c^2p^2 + m^2c^4} + \frac{m\omega^2x^2}{2}$$

Expand this expression in powers of p/mc and compute the lowest order relativistic correction $\delta\omega$ to the frequency ω of the ordinary (non-relativistic) hormonic oscillator as a function of the system's energy E. (Hint: Introduce action-angle variables, solve the 0^{th} order problem, and the use the 1^{st} order classical perturbation theory.

Quantum Mechanics Qualification
Fall, 2022

Problem 1 Answer the following questions *briefly*
(a) 16% Explain the following terms briefly: (i) canonical momentum and mechanical momentum (ii) spontaneous emission (iii) collapse of state (iv) Kramer degeneracy
(a) 6% What are generators of translation?
(b) 6% Express the Hermitian conjugate of $\hat{O}$ in terms of $\hat{O}$ with $\hat{O}$ being (i) $3x\frac{d}{dx}$ and (ii) $[A, B]$, where A and B are observables.
(c) 6% Express the probability density operator in the momentum space $\hat{\rho}(p)$ in terms of p and $\hat{\mathrm{p}}$ such that $\langle\Psi|\hat{\rho}(p)|\Psi\rangle = |\Psi(p)|^2$. Here $\Psi(p) = \int \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\Psi(x)$ is the Fourier transformation of $\Psi(x)$.
(e) 6% Consider a particle whose position is denoted by $\vec{r} = (x, y, z)$ and total angular momentum is denoted by $\vec{J}$. Find $\exp(iJ_z\phi/\hbar)\, x^2 \exp(-iJ_z\phi/\hbar)$ in terms of x and ϕ.

Problem 2 Consider a particle of mass m that moves freely in one dimension. Suppose that initially, the wavefunction of the particle at $t = 0$ is

$$\psi(x, 0) = (\pi\Delta^2)^{-1/4} \exp(ik_0x - x^2/2\Delta^2),$$

(a) 6% Find the position operator in the Heisenberg picture $\hat{\mathrm{x}}_H(t)$ (for $t > 0$) in terms of the operators $\hat{\mathrm{x}}$ and $\hat{\mathrm{p}}$ defined in the Schrodinger's picture. Calculate the commutator $[\hat{\mathrm{x}}_H(t), \hat{\mathrm{x}}_H(t')]$.
(b) 8% By solving appropriate equations of motions, find $\Delta x(t)$ and $p^2(t)$ for $t > 0$.
(c) 6% Suppose that the particle carries a charge q. Under the influence of a time-dependent electric field $E\cos\omega t$, the particle is governed by the Hamiltonian $\hat{\mathrm{H}} = \hat{\mathrm{p}}^2/2m + \hat{\mathrm{V}}$ with $\hat{\mathrm{V}} = -\mathrm{qE}\cos(\omega \mathrm{t})\hat{\mathrm{x}}$. Solve $\mathrm{x}_H(t)$and $\mathrm{p}_H(t)$ in terms of $\hat{\mathrm{p}}$ and $\hat{\mathrm{x}}$ (defined in the Schrodinger picture).

Problem 3
Consider the coupling of three spin-1/2 particles. Let $|\alpha\beta\gamma\rangle$ denote the state when the first particle in ithe state $|\alpha\rangle$, the 2nd in $|\beta\rangle$ and the 3rd in $|\gamma\rangle$, where α, β, and γ are either $+$ (spin up) or $-$ (spin down).
(a) 7% Construct all states with total angular momentum $J = 3/2$.
(b) 8% Construct all states with $J = 1/2$. (Hint: add two spin-1/2 partciles first, and then include the 3rd particle.)

Problem 4 10%
A particle of mass m is described by the wave function

$$\psi_E(r, \theta, \phi) = A\exp(-r/a_0), \tag{1}$$

where r, θ, and ϕ are spherical coordinates and a_0 is a positive constant. Assuming that ψ_E is an energy eigenstate in a central potential $V(r)$ and $V(r) \to 0$ as $r \to \infty$, find the energy E of this state, $V(r)$, and all other possible energies for this particle.

Problem 5
Consider a perturbation $H' = \alpha x^4$ to the motion of the harmonic oscillator. Here α is a positive number. Let m and ω be the mass and the natural frequence of the oscillator. Answer the following questions:
(a) 7% Show that the first-order correction to the nth unperturbed eigen-energies are

$$E_n^1 = (3\hbar^2\alpha)/(4m\omega^2)[1 + 2n + 2n^2]. \tag{2}$$

No matter how small α is, the perturbation expansion will break down for some large enough n. Why?
(b) 8% If we make the perturbation time-dependent, $H' = \alpha x^4 e^{-t^2/\tau^2}$, between $t = -\infty$ and $t = +\infty$. What is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = +\infty$? You might find the following identity useful. $\int_{-\infty}^{\infty} dxe^{-ax^2+bx} = \sqrt{\pi/a}e^{b^2/4a}$.

Qualifying Examination – Statistical Mechanics

Sep 17-18, 2022

Please explain the logic behind your answers.

Problem 1. *Bosons in Harmonic traps(30 points):* Let us consider a particle in the anisotropic harmonic-oscillator potential $V(\boldsymbol{r}) = \frac{1}{2}\left(K_x x^2 + K_y y^2 + K_z z^2\right)$. Therefore, we can consider the system as three independent harmonic oscillators in three different directions x, y and z. The corresponding energy levels can be parameterized by three non negative integers (n_x, n_y, n_z) as

$$E(n_x, n_y, n_z) = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z. \tag{1}$$

1. Let us consider n_i are continuous variables and neglect the zero-point energies (The $\frac{1}{2}\hbar\omega_i$ in $E(n_x, n_y, n_z)$.), we can simplify the energy as $E \approx \varepsilon_x + \varepsilon_y + \varepsilon_z$ where $\varepsilon_i = n_i\hbar\omega_i$. If we define $G(E)$ as the number of states below energy a specific energy E. Derive the expression of $G(E)$ in terms of E, $\omega_x, \omega_y, \omega_z$ and $\hbar$. (Hint: There are several way to derive this result. One approach is to consider the problem in the (n_x, n_y, n_z) space. What is the geometric meaning of the states below energy E?)

2. Another way to understand density of state is $g(E) = \frac{dG(E)}{dE}$. Use the above expression to get the density of states.

3. We can evaluate the particles in the excited states using

$$N_{ex} = \int_0^\infty dE g(E) \frac{1}{e^{E/k_B T} - 1}. \tag{2}$$

Here, we assume N_{ex} reaches its maximum for $\mu = 0$. The definition of the transition temperature, T_c, of Bose-Einstein condensation is when the total number of particles can be just accommodated in excited states. That is,

$$N = N_{ex}(T = T_c, \mu = 0) = \int_0^\infty dE g(E) \frac{1}{e^{E/k_B T} - 1}. \tag{3}$$

$\int_0^\infty dx \frac{x^{\alpha-1}}{e^x - 1} = \Gamma(\alpha)\zeta(\alpha)$. Here, $\zeta(\alpha) = \sum_{n=1}^\infty n^{-\alpha}$ is the Riemann zeta function and $\Gamma(\alpha)$ is the gamma function. Derive the transition temperature of this system and express the result using $N, \omega_x, \omega_y, \omega_z$ and the Riemann zeta function.

Problem 2. *Ising model(15 points):* Considering the Ising model $H = -J\sum_{\langle i,j\rangle} \sigma_i\sigma_j$ where $\sigma_i = \pm 1$ and $\langle i, j\rangle$ denotes the nearest-neighbor pairs of sites.

1. For a system of coordinate number z, use mean-field approximation to find the critical temperature T_c below which spontaneous magnetization exists.

2. Show that the magnetic susceptibility $\chi \propto (T - T_c)^{-1}$ at $T \gg T_c$
3. Use entropy argument to show that in 1D there is no phase transition, i.e., $T_c = 0$.

Problem 3. *Free fermion problem(10 points):* Let the density of states of the electrons in a system be assumed to be a constant D for $\varepsilon > 0$ ($D = 0$ for $\varepsilon < 0$). Here, ε represents the energy. We assume the total number of electrons be equal to N. Here $N = D \int_0^\infty \frac{1}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon$.

1. Calculate the chemical potential μ at $T = 0K$.
2. When the system is in the regime of quantum degeneracy, estimate or derive the expression of the specific heat and show that it is proportional to T.

Problem 4. *Two-level system(15 points):* Consider a system of N distinguishable particles, which have two energy levels, $E_0 = -\mu B$ and $E_1 = \mu B$, for each particles. Here μ is the magnetic moment and B is the magnetic field. The particles populate the energy levels according to the classical distribution law.

1. Calculate the average energy of such system at temperature T
2. Calculate the specific heat of the system
3. Calculate the magnetic susceptibility

Problem 5. *Random walk(10 points):* A drunkard is kicked out of the pub. Being fully intoxicated, he walks forward and backward with equal probability. According to the senior high school math, the probability of finding him at position n after N step is $P(n, N) = C_{N_\rightarrow}^{N} \left(\frac{1}{2}\right)^{N_\rightarrow} \left(\frac{1}{2}\right)^{N_\leftarrow}$ where $N_\rightarrow$ and $N_\leftarrow$ represent the number of steps forward and backward, respectively. By definition, we have $N = N_\rightarrow + N_\leftarrow$ and $n = N_\rightarrow - N_\leftarrow$. Assume $N \gg n \gg 1$ so that the Stirling formula can be used to approximate all large factorials: $\lim\limits_{N \gg 1} N! \approx N \ln N - N$. Show that $P(n, N)$ can be reduced to the Gaussian distribution : $P(n, N) \sim \frac{1}{\sqrt{N}} exp\left[-\frac{n^2}{4N}\right]$.

Problem 6. *Spins and vacancies on a surface and the negative temperature(20 points):* Consider a surface with N sites and a collection of non-interacting spin $\frac{1}{2}$ particles. For each site the energy $\varepsilon = 0$ if there is a vacancy and $\varepsilon = -W$ if there is a particle present, where $-W < 0$ is the binding energy.

1. Let $N_\pm$ be the number of particle with spin $\pm\frac{1}{2}$, $Q = N_+ + N_-$ be the number of spins, N_0 be the number of vacancies, $M = N_+ - N_-$ be the surface magnetization. We have $N_+ + N_- + N_0 = N = Q + N_0$. In the microcanonical ensemble, compute the entropy $S(Q, M)$. (Hint: Calculate the number of states available to the system and express N_+, N_-, N_0 with Q and M.
2. Let $q = \frac{Q}{N}$ be the dimensionless particle density and $m = \frac{M}{N}$ be the dimensionless magnetization density. Assuming that we are at the thermodynamic limit where N, Q and M all tend to infinity, but with q and m finite. Find the temperature $T(q, m)$. (Hint: Use Stirling's formula $\ln N! \approx N \ln N - N$.)
3. Show explicitly that T can be negative for this system. What does negative T mean? What physical degrees of freedom have been left out that would avoid this strange property?