Qualifying Examination - Quantum Mechanics

Fall, 2020

1. (Terminology explanation, 5 points each, 20 points in total)
   Please explain the following terminologies in QM (half page each)
   (a) Connection between symmetry and conservation law,
   (b) WKB approximation,
   (c) Fermi-Golden Rule,
   (d) Dirac Equation

2. (Propagation in 1D, 10 points each, 20 points in total)
   (a) Calculate the propagator in coordinate space of a free moving particle in 1D free space with mass *m*, i.e. $\left\langle x|\hat{U}(t,t')|x'\right\rangle$. (b) For an initial wavefunction, $\phi(x,0)=C\exp(-x^2/a^2)$, calculate its wavefunction at time *t*.

3. (Spin, 5 points each, 20 points in total)
   Consider an electron in spin state $|S_x=1/2\rangle$, was injected with a velocity *v* into $x\geqq 0$ regime at *t*=0, where there is a magnetic field (*B*) in the z direction (no magnetic field for *x*<0). (a) Write down the spin state in the basis of eigenstate in *z* direction. (b) Assuming the spin coupling with magnetic field as $H=-\gamma\mathbf{S}\cdot\mathbf{B}$, write down the equation of motion for the coefficients of the spin state in the presence of magnetic field. (d) What is the spin state after exiting the regime at *x*=*L* ? (e) When the probability if measuring the spin polarization in y direction after that ?

4. (Approximation Method, 10 points each, 20 points in total)
   Consider a simple harmonic oscillator, $H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2$, with a perturbation, $H'=\alpha x^4$. Show that the first-order correction to the unperturbed eigen-energies are $E_n^1=\frac{3\hbar^2\alpha}{4m^2\omega^2}\left[1+2n+2n^2\right]$. Using variational wavefunction, $\phi(x)\equiv\frac{1}{\sqrt{\sqrt{\pi}R}}\exp(-x^2/R^2)$, derive the equation or evaluate *R* to determine the most possible ground state energy. (Hint: $\int_0^\infty x^{2n}e^{-\frac{x^2}{a^2}}\,dx=\sqrt{\pi}\frac{a^{2n+1}(2n-1)!!}{2^{n+1}}$)

5. (Scattering, 10 points each, 20 points in total)

A point particle of mass m and incident energy E is scattering off the potential $V(r) = V_0 \exp(-r^2/R^2)$. (a) Calculate scattering amplitude, $f(\theta)$, within the first Born approximation. (b) Derive the s-wave scattering length in the limit of zero energy.

Classical Electrodynamics (2020 September)
Please use SI unit system. If not, please indicate the system you use.

1. Explain the following terms.
(a) Green's theorem and Green's function (8%)
(b) Lorentz gauge and Coulomb gauge (8%)
(c) Group and phase velocities (8%)
(d) Plasma frequency and skin depth (8%)
(e) Poynting's theorem and the conservation of energy (8%)

2. Derive the conservation of charge from the Maxwell equations. (10%)

3. What is the Maxwell stress tensor? Express the conservation of momentum with the Maxwell stress tensor. (10%)

4. If $\mathbf{E}$ and $\mathbf{B}$ are perpendicular in the laboratory and $E = 2B$, can you find a reference frame such that the field is pure electric or magnetic? If yes, what is the velocity of the reference frame relative to the laboratory? (10%)

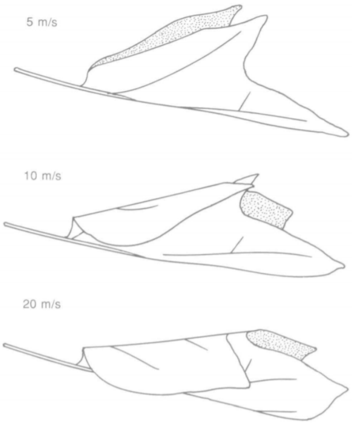
5. Consider an infinite long hollowed iron bar of inner and outer radii a and b, respectively, carries a uniform magnetization, $\mathbf{M} = M_0 \hat{z}$. Find the bound surface currents $\mathbf{K}_b$ on the inner and outer surfaces. (10%)
[Hint: use cylindrical coordinate, (ρ, ϕ, z) .]

6. Two concentric conducting shells of inner and outer radii a and b ($b>a$), respectively. The inner shell is connected to a potential $V(a,\theta)$ (to be given), while the outer shell is grounded $V(b,\theta) = 0$.
(a) If $V(a,\theta) = V_0$ (constant), find the potential at $r<a$, $a<r<b$, and $r>b$. (10%)
(b) If $V(a,\theta) = V_0 \sin^2 \theta$, find the potential everywhere between the shells ($a<r<b$). (10%)
[Hint: use Legendre polynomials $P_0(x) = 1,\ P_1(x) = x,$ and $P_2(x) = (3x^2 - 1)/2$].

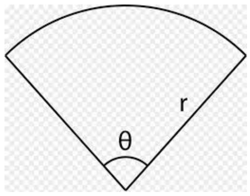
Qualification examination (September 2020)

- Classical Dynamics (20 points each) -

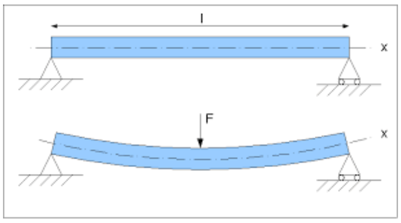
1. (Optional, bonus) Famous biomechanist Steven Vogel[1] found that leaves tend to curl up into a cone when blown by the wind. Use v, ϕ to denote the wind speed and apex angle of the cone(圓錐頂角). For simplicity, assume the collisions between air particles and the leaf to be elastic, and the leaf is of fan-shape(扇形) with radius r and angle θ before being blown by wind.



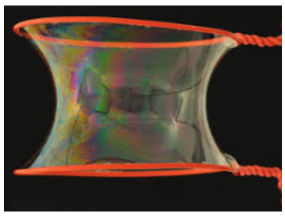
   (a) Find the torque exerted by the wind on a complete cone. Note the surface area of cone is smaller than that of leaf.



   (b) We expect the cone to become sharper when the wind gets stronger, namely, $\phi(v)$ ought to be a decreasing function of v. But, how do we solve $\phi(v)$? Given that if a plate of length ℓ, width a, and thickness t is bended with curvature[2] $1/R$, the bending energy equals $Yt^3\ell a/R^2$ where Y is the Young's modulus of plate. First, use the above information to find the total potential energy E_b stored on the leaf. Then, differentiate E_b by ϕ to determine the restoring torque. Finally, solve for $\phi(v)$ by balancing this torque with the external torque from wind.



2. (Catenary) (a) Soap film is formed between two circular wires of radius R that are separated by distance L. Use the calculus of variations to find the shape of soap film. Neglect the gravity. Show that there is an upper limit to L beyond which the soap film will burst.



   (b) Blowing soap bubbles is a common childhood experience. The faster we blow, the smaller bubbles are, and the more their number. Try to build a model and predict how the bubble radius r varies with the air speed v that you blow.

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[1] S. Vogel, *Drag and Reconfiguration of Broad Leaves in High Winds*, J. Exper. Bot. **40**, 941 (1989).

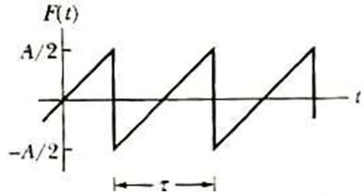
[2] You can imagine R as the radius of an imagined circle that can fit the bended surface of the plate.

3. A block of mass m with velocity v_0 slides on a horizontal surface before hitting and climbing a sledge of mass M and with inclination angle θ. Assume the surface is smooth, but there exists a dynamic friction with coefficient μ between the block and the sledge.

   (a) What is the initial velocity v_i of the block soon after it mounts on the sledge?

   (b) Find the maximum vertical height the block can reach on the sledge.

4. Find the current $I(t)$ on an RLC-circuit that is driven by a saw-tooth voltage $F(t)$. Assume no current or charge at $t = 0$. *Hint*: $I(t)$ consists of two parts, temporary and long-term solutions.



5. Find the displacement $y(x,t)$ for a "plucked string," where the midpoint of the string (of length L) is pulled up by a displacement of h (such that the string assumes a triangular shape) and then release from rest. Use ρ, T to denote the mass density and tension of the string. *Hint*: $y(x,t)$ obeys the wave equation $\frac{\partial^2 y}{\partial^2 t} = \frac{T}{\rho}\frac{\partial^2 y}{\partial^2 x}$.

6. (Special relativity) An electromagnetic wave propagates with velocity $\vec{u} = \frac{c}{n}(\cos\theta, \sin\theta)$ in a stationary medium of refractive index n.

   (a) Find $\vec{u}'$ when the inertia frame is boosted with speed v in the x-direction. Use $|\vec{u}'| = \frac{c}{n'}$ to determine the refractive index n' of a moving medium. Note that the formula for n' is dependent of the angle θ.

   (b) Prove that the relativistic energy E and momentum $\vec{P}$ are a 4-vector, namely, they obey the same Lorentz transformation as the space-time. Use it to derive the longitudinal and transverse optical Doppler effect.

   (c) (Optional) Now let's repeat the goal of (a) by using $\vec{P}$ instead of $\vec{u}$ via the relation, $\frac{E}{P} = \frac{\hbar\omega}{\hbar k} = u = \frac{c}{n}$. First, use (b) to find E', P'. Dividing them then gives n' via $\frac{E'}{P'} = \frac{c}{n'}$. You will find this result to be different from that of (a). Why? And which result is more correct?

Qualifying Examination – Statistical Mechanics

Fall, 2020

1. 20 points (micro-canonical, canonical, and grand-canonical ensembles) Lagrange multipliers allow one to find the extremum of a function $f(x)$ given a constraint $g(x) = g_0$. One sets to zero the derivative of $f(x) + \lambda(g(x) - g_0)$ with respect to x. Let us now use Lagrange multipliers to find the maximum of the entropy $S = -k_B \sum_i p_i \ln(p_i)$ by constraining
   (a) the normalization $(\sum_i p_i = 1)$,
   (b) the energy $(\langle E\rangle = \sum_i E_i p_i)$,
   (c) the number $(\langle N\rangle = \sum_i N_i p_i)$,

   to obtain the distribution function p_i for micro-canonical, canonical, and grand-canonical ensembles, respectively.

2. 15 points (black hole thermodynamics) Astrophysicists have long studied black holes: the end state of massive stars which are too heavy to support themselves under gravity. As the matter continues to fall into the center, eventually the escape velocity reaches the speed of light. After this point, the in-falling matter cannot ever communicate information back to the outside. A black hole of mass M has radius $R_s = G\frac{2M}{c^2}$ where G is the gravitational constant and c is the speed of light. By combining methods from quantum mechanics and general relativity, Hawking calculated the emission of radiation from a black hole and found it to follow the perfect black-body radiation at a temperature $T = \frac{\hbar c^3}{8\pi G M k_B}$. According to Einstein's theory, the energy of the black hole is $E = Mc^2$.
   (a) Calculate the specific heat of the black hole.
   (b) Calculate the entropy of the black hole, by using the definition of temperature $\frac{1}{T} = \frac{\partial S}{\partial E}$ and assuming the entropy is zero at mass $M = 0$. Express your result in terms of the surface area $A = 4\pi R_s^2$, measured in units of the Planck length $L^* = \sqrt{\hbar G/c^3}$.

3. 15 points (three-spin interaction) Consider a 1D system of N spins with the Hamiltonian: $H = -J\sum_{i=1}^{N-2} S_i S_{i+1} S_{i+2}$, where $S_i = \pm 1, i = 1 \ldots N$ being the spin states and J is a constant. Assume open boundary conditions (**NOT** periodic!).
   (a) Calculate the canonical partition function and the Helmholtz free energy F.
   (b) Calculate the free energy per site in the thermodynamic limit.

   Hint: The transfer-matrix method is not necessary for this problem.

4. 25 points (2D fermions) Consider an ideal gas of N spin-1/2 fermions at zero temperature confined to an area A in 2D. The fermions are in an external magnetic field H. The energy of a particle is $\epsilon = \frac{p^2}{2m} \pm \mu_B H$, where μ_B is the Bohr magneton.

   (a) Give an expression for the chemical potential μ_0 for vanishing magnetic field as a function of the particle density N/A.

   (b) Calculate the average particle energy as a function of μ_0 for weak external magnetic fields. (Up to second order in H.)

   (c) Calculate the susceptibility $\chi = \partial m/\partial H$ for weak external magnetic fields.

5. 25 points (adsorption) Consider an ideal gas (temperature T, chemical potential μ) in contact with a surface with N adsorption sites. Each site may be occupied by 0, 1, or 2 gas molecules. The energies of a site with 0, 1, 2 adsorbed molecules are $0, -\epsilon, -3\epsilon/2$ respectively. There is no interaction between molecules at different adsorption sites.

   (a) Calculate the grand canonical partition function for a fixed number N of adsorption sites.

   (b) Use the grand canonical partition function to derive the mean number of the adsorbed particles per site $\langle n \rangle$ and the mean internal energy per site $\langle u \rangle$ as a function of $T, \mu,$ and ϵ.

   (c) For $T = 0$ sketch $\langle n \rangle$ for a constant μ as a function of ϵ.

   (d) Calculate $\langle n \rangle$ for large temperature. (No corrections in T are necessary.)