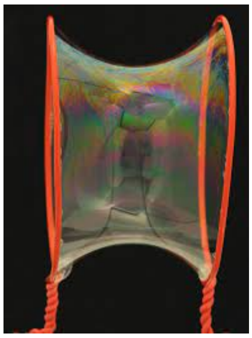
Qualifying Examination for Classical Dynamics Fall, 2021

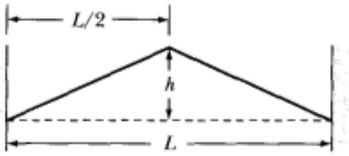
1. (Calculus of variations, 20 points)

   (a) A soap film is built between two circular wires of radius R that are separated by a distance L. Air is allowed to flow through the wires. Please (i) find the shape of film, and (ii) show that there exists an upper bound for L beyond which the film becomes unstable.

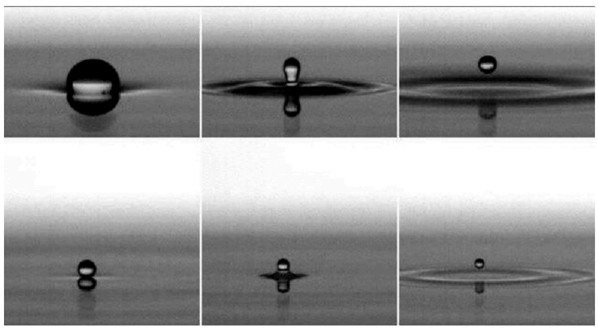


   (b) Attach these two circular wires to a soap bubble of radius $2R$ before pulling them apart to a distance $L > 2\sqrt{3}R$. (iii) Please show how its shape will differ from that of the open film in (a). Note that the volume of air inside the bubble is roughly conserved.

2. (Vibrating string, 20 points) A string, composing of two segments with mass per unit length $\rho_{1,2}$ and length $L_{1,2}$ where $L_1 + L_2 = L$, is fixed on both ends and under tension T. Please find (i) condition for wavelength $\lambda_{1,2}$ of standing waves and (ii) displacement $q(x,t)$ if the string is plucked in the middle by a height h at $t=0$ and released from rest. *Hint:* The wave equation is $\frac{d^2 q_{1,2}}{dt^2} = \frac{T}{\rho_{1,2}} \frac{d^2 q_{1,2}}{dx^2}$.

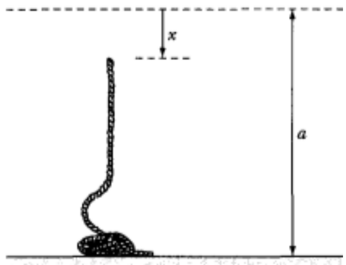


3. (Rocket motion, 10 points) When a droplet of water falls from a medium height on the surface of water, it does not get absorbed straight away. Underneath the droplet is a very thin layer of air, which allows it to sit temporarily above the surface. After the air layer is pushed aside by the droplet's weight, the surface of water pinches at water for a split second absorbing it. However, it happens so fast that the connection is cut off between them and a smaller droplet, which hasn't been absorbed bounces

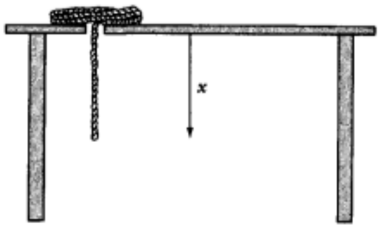


up. Please treat this phenomenon[1] like a rocket motion and estimate the ratio of droplet size after and before the cascade. *Hint*: The water released by droplet acts like the exhaust to propel the remaining droplet into the air.

4. (Rope problem, 20 points)



(a) A rope of mass per unit length ρ and length a suspended just above a table. If it is released from rest at the top, find the force on the table when a length x of the rope has dropped on the table.



(b) A smooth rope is places above a hole on a table. One end of rope falls through the hole at $t = 0$, pulling steadily on the rest of rope. Find the velocity and acceleration of the rope as a function of the distance to the end of rope, x.

5. (Central-force motion, 10 points) Please show that the orbit of planets belongs to the mathematical set of quadratic curve or a conic section.

6. (Special theory of relativity, 10 points) Two trains of proper length $L_{1,2}$ and speed $v_{1,2}$, as measured by the stationary frame, enter the station from opposite directions. Please find the time it takes for the trains to pass each other[2] as observed by (i) the conductor of train 1 and (ii) the passenger on the platform. (iii) Check whether your answers in (i) and (ii) obey the formula for time dilation. If not, explain why.

7. (Saxon bowl, 10 points) The Saxons placed a bowl with a hole in its bottom in water and used the time it took the bowl to submerge to limit orations. According to history, a famous Greek prostitute also used such bowls to allocate her customers' time. Please find the relationship between the radius of hole and the time till submergence.

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[1] https://thekidshouldseethis.com/post/12162224250 has a nice video for coalescence cascade

[2] The event is defined to start when the fronts of trains meet, and end when their rears cross.

Qualifying Examination - Quantum Mechanics

Fall, 2021

1. (Terminology explanation, 30 pts in total, 5pts each)
   Please explain the following terminologies in QM (0.5 page each, including words, equations, and figures if the latter is necessary)
   (a) Propagator of a free particle
   (b) Feynman Path Integral
   (c) WKB method
   (d) Identical particles in quantum mechanics
   (e) Aharonov-Bohm effect
   (f) Scattering length

2. (Variational method, 10+10 points)
   (a) Consider a 1D attractive potential, $V(\mathrm{x}) = -V_0$ for $|x| < a$. Show that there must be at least one bound state for any value $V_0 > 0$ by using variational method. (b) Explain why this method does not apply in 3D case (i.e. in 3D, V_0 has to be larger than a certain value in order to have a bound state possible.

3. (Electron spin, 5+5+10 points)
   An electron of spin $\left|S_x = \frac{1}{2}\right\rangle$ is injected with velocity $(v,0,0)$ into $x \geq 0$ regime where there is a magnetic field $(0,0,B)$ (while $B = 0$ elsewhere).
   (a) Assuming the electron follows a classical trajectory, what will be the radius of its trajectory ? What is the time, T, it will need to eventually exist the regime of finite magnetic field ?
   (b) Write down the Hamiltonian to describe the spin degree of freedom (quantum mechanically) in the co-moving frame.
   (c) What is its spin state after time, T ?

4. (Perturbation theories, 10+5+10 points)
   Consider a perturbation $H' = \alpha x^4$ to the harmonic oscillator problem.
   (a) Show that the first-order correction to the unperturbed eigen-energies are $E_n^1 = \frac{3\hbar^2\alpha}{4m^2\omega^2}\left[1 + 2n + 2n^2\right]$.
   (b) Argue that no matter how small α is, the perturbation expansion will break down for some large enough n. What is the physical reason?
   (c) If we are careful with how the perturbation is turned on; say, $H' = \alpha x^4 e^{-t^2/\tau^2}$ between $t = -\infty$ and $t = \infty$, what is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = \infty$?

Qualifying Examination – Statistical Mechanics

Sep 25-26, 2021

Please explain the logic behind your answers.

Problem 1. *Bosons in Harmonic traps(30 points):* Let us consider a particle in the anisotropic harmonic-oscillator potential $V(\boldsymbol{r}) = \frac{1}{2}\left(K_x x^2 + K_y y^2 + K_z z^2\right)$. Therefore, we can consider the system as three independent harmonic oscillators in three different directions x, y and z. The corresponding energy levels can be parameterized by three non negative integers (n_x, n_y, n_z) as

$$E(n_x, n_y, n_z) = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z. \tag{1}$$

1. Let us consider n_i are continuous variables and neglect the zero-point energies (The $\frac{1}{2}\hbar\omega_i$ in $E(n_x, n_y, n_z)$.), we can simplify the energy as $E \approx \varepsilon_x + \varepsilon_y + \varepsilon_z$ where $\varepsilon_i = n_i\hbar\omega_i$. If we define $G(E)$ as the number of states below energy a specific energy E. Derive the expression of $G(E)$ in terms of E, $\omega_x, \omega_y, \omega_z$ and $\hbar$. (Hint: There are several way to derive this result. One approach is to consider the problem in the (n_x, n_y, n_z) space. What is the geometric meaning of the states below energy E?)

2. Another way to understand density of state is $g(E) = \frac{dG(E)}{dE}$. Use the above expression to get the density of states.

3. We can evaluate the particles in the excited states using

$$N_{ex} = \int_0^\infty dE g(E) \frac{1}{e^{E/k_BT} - 1}. \tag{2}$$

   Here, we assume N_{ex} reaches its maximum for $\mu = 0$. The definition of the transition temperature, T_c, of Bose-Einstein condensation is when the total number of particles can be just accommodated in excited states. That is,

$$N = N_{ex}(T = T_c, \mu = 0) = \int_0^\infty dE g(E) \frac{1}{e^{E/k_BT} - 1}. \tag{3}$$

   $\int_0^\infty dx \frac{x^{\alpha-1}}{e^x - 1} = \Gamma(\alpha)\zeta(\alpha)$. Here, $\zeta(\alpha) = \sum_{n=1}^\infty n^{-\alpha}$ is the Riemann zeta function and $\Gamma(\alpha)$ is the gamma function. Derive the transition temperature of this system and express the result using $N, \omega_x, \omega_y, \omega_z$ and the Riemann zeta function.

Problem 2. *A simplified model of hemoglobin(10 points):* Hemoglobin is a protein in the blood that carries oxygen from the lungs to the muscles. It is formed by four units. Each unit can carry an O_2 molecule or not. For a hemoglobin, there will be 2^4 configurations as shown in Fig.1. (Each unit has two choices, carries nothing or an O_2 molecule. Therefore we have 2^4 possible choices.)

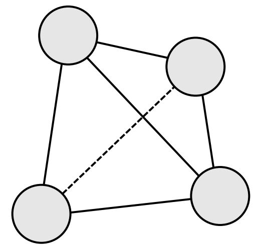

Figure 1: Schematic model of hemoglobin. The gray circles represent the unit that can carry an O_2 molecule. We assume the four units form a tetrahedral network structure. We can approximate the hemoglobin using a simplified model: The energy gain for an O_2 to bind with a unit is $-\varepsilon_0$. If two nearby units both contain O_2, the energy can further reduced by energy $-J$.

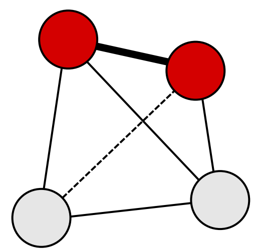

Figure 2: One particular configuration of hemoglobin. The red circles represent the units that are occupied by O_2 molecules. The thick line represent the additional energy reduction, $-J$, for nearby units are occupied by O_2 molecules. There will be other 5 configurations with equal energy.

As described in the caption of Fig.1, we can model the hemoglobin by assigning the binding energy $-\varepsilon_0$ and the additional gain, $-J$, if two nearby unit both bind with O_2(We try to pack O_2 to our hemoglobin as much as possible, a simple minded physicist guess). For example, considering the configuration in Fig. 2, the total energy of the simplified model is $(-\varepsilon_0) \times 2 + (-J) \times 1$. Consider we have $N/4$ hemoglobin, labeled by $\alpha = 1 \sim \frac{N}{4}$. On each hemoglobin, we have 4 units which can bind with O_2 molecules. We can use $\tau_{\alpha,i} = 0, 1$ for $i = 1 \sim 4$ to denote whether the i-th unit on the α-th hemoglobin is occupied by an O_2 molecule or not.

1. Write down the expression of the grand canonical partition function of the system with $\frac{N}{4}$ hemoglobin. Express the result using $x = exp\left[\frac{\varepsilon_0+\mu}{k_BT}\right]$and $y = exp\left[\frac{J}{k_BT}\right]$. (Hint: we consider the molecules of hemoglobin to be independent. So, we can factorize the partition function of the system into products of partition functions of individual hemoglobin molecule. Then, within each Hemoglobin, we can enumerate all the possible configurations and the corresponding energy to construct our partition function. Be careful about the number of configurations with equal energy. The degeneracy of energy in this system is not very regular, so one needs to specify them explicitly.)

2. Evaluate the average number $\langle M\rangle = \sum_{\alpha=1}^{N/4}\sum_{i=1}^{4}\langle\tau_{\alpha,i}\rangle$ of adsorbed molecules as a function of x, y.

Problem 3. *Free fermion problem(10 points):* Let the density of states of the electrons in a system be assumed to be a constant D for $\varepsilon > 0$ ($D = 0$ for $\varepsilon < 0$). Here, ε represents the energy. We assume the total number of electrons be equal to N. Here $N = D\int_0^\infty \frac{1}{e^{\beta(\varepsilon-\mu)}+1}d\varepsilon$.

1. Calculate the chemical potential μ at $T = 0K$.

2. When the system is in the regime of quantum degeneracy, estimate or derive the expression of the specific heat and show that it is proportional to T.

Problem 4. *Two-level system(20 points):* Consider a system of N distinguishable particles, which have two energy levels, $E_0 = -\mu B$ and $E_1 = \mu B$, for each particles. Here μ is the magnetic moment and B is the magnetic field. The particles populate the energy levels according to the classical distribution law.

1. Calculate the average energy of such system at temperature T
2. Calculate the specific heat of the system
3. Calculate the magnetic susceptibility

Problem 5. *Simple Harmonic oscillator(15 points):* Consider a 1D quantum simple harmonic oscillator with ground state energy $\varepsilon_0 = \frac{\hbar\omega}{2}$, in thermal equilibrium:

1. Find the canonical partition function
2. Find the free energy F
3. Find the average energy U from F
4. Find the average number of excitation $\langle n \rangle$

Problem 6. *Random walk(15 points):* A drunkard is kicked out of the pub. Being fully intoxicated, he walks forward and backward with equal probability. According to the senior high school math, the probability of finding him at position n after N step is $P(n,N) = C^N_{N_\rightarrow} \left(\frac{1}{2}\right)^{N_\rightarrow} \left(\frac{1}{2}\right)^{N_\leftarrow}$ where $N_\rightarrow$ and $N_\leftarrow$ represent the number of steps forward and backward, respectively. By definition, we have $N = N_\rightarrow + N_\leftarrow$ and $n = N_\rightarrow - N_\leftarrow$. Assume $N \gg n \gg 1$ so that the Stirling formula can be used to approximate all large factorials: $\lim_{N\gg 1} N! \approx N \ln N - N$. Show that $P(n,N)$ can be reduced to the Gaussian distribution : $P(n,N) \sim \frac{1}{\sqrt{N}} exp\left[-\frac{n^2}{4N}\right]$.

Classical Electrodynamics (2021 September)

Please use SI unit system. If not, please indicate the system you use.

1. Explain the following terms as clear as possible. (8*5%)

(a) Cherenkov radiation

(b) Synchrotron radiation

(c) Group and phase velocities

(d) Liénard-Wiechert potentials

(e) Complex Poynting's theorem

(f) Plasma frequency and skin depth

(g) Lorentz gauge and Coulomb gauge

(h) Perfect conductor and super conductor

2. The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right),$$

where q is the magnitude of the electronic charge, and $\alpha = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your results physically. (10%)

3. Derive the Green theorem step-by-step.

(a) Green's first identity and second identity. (5%)

(b) Application of the Green theorem to electrostatic boundary-value problems with the Green function satisfying $\nabla'^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ with $G_D(\mathbf{x},\mathbf{x}') = 0$ for $\mathbf{x}'$ on S. (10%)

4. A hollow rectangular waveguide has a cross section of $a \times b = 7.112$ mm$\times$3.556 mm.

(a) Find the electric and magnetic fields for the dominant mode TE_{10}. (5%)

(b) Estimate the cutoff frequencies for the first three modes (TE_{10}, TE_{20}, TE_{01}). (5%)

(c) Qualitatively plot the dispersion relation ($\omega - k_z$ diagram) of the dominant mode. (5%)

5. Prove that $E^2 - B^2$ and $\mathbf{E}\cdot\mathbf{B}$ are invariant under the Lorentz transformation. (20%)