Qualifying Examination - Quantum Mechanics

Spring, 2021

1. (Terminology explanation, 30 pts in total, 5pts each)
   Please explain the following terminologies in QM (0.5 page each, including words, equations, and figures if the latter is necessary)
   (a) Uncertainty principle between any two physical quantities
   (b) Propagator
   (c) Fermi golden rules
   (d) Bosons and Fermions
   (e) Spin-orbital coupling
   (f) Rabi oscillation

2. (Delta-function potential, 25 pts in total and 5 pts each)
   (a) Find the ground state energy for a 1D attractive delta-function potentials $V(x) = -\alpha \cdot \delta(x)$ for a particle of mass *m*.
   (b) Write down the equations (together with the proper boundary condition and continuity equations) for the eigenstate of a double delta-trapping potential: $V(x) = -\alpha \cdot \delta(x - L/2) - \alpha \cdot \delta(x + L/2)$.
   (c) Assuming *L* is large, draw a schematic plot of the ground state and the first excited state wavefunctions.
   (d) What is the length scale, L_0, (in terms of mass *m* and α) for *L* such that the ground state energy of the above double delta-trapping potential is almost the same as the ground state energy of a single delta potential as $L > L_0$? explain why.
   (e) Draw a schematic plot to show how the ground-state and the first excited state energy change as a function of L, but there is no need to solve for its exact value except at the asymptotic cases of $L = 0$ and $L = \infty$.

3. (Electron Spin, 15 pts in total, 5 pts each)
   Consider a spin 1/2 electron in the following situation:
   (a) One first measure the spin along the *z*-axis using a Stern-Gerlach apparatus, and define the state to be a state of eigenvalue, $\hbar/2$. What is the probability that a subsequent measurement of the spin in the direction $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ still yields $+\hbar/2$?
   (b) After the measurement of the electron's spin along the axis $\hat{n}$ with the eigenvalue $+\hbar/2$, calculate the probability that a subsequent measurement of the spin along the *x*-axis yields $-\hbar/2$?
   (c) After above measurement, now applying a magnetic field, **B**, in the *z* axis, what will the wavefunction be after a time *t* ? (assuming the Hamiltonian of spin with magnetic field is $H = -\gamma \mathbf{S} \cdot \mathbf{B}$)

4. (Scattering, 10 pts in total, 5pts each)
   A point particle of mass m and incident energy E is scattering off the potential $V(\mathrm{r}) = ge^{-r^2/R^2}$.
   (a) Calculate the first Born approximation to the differential cross sections.
   (b) Calculate the s-wave scattering length in the zero energy limit.

5. (Time-dependent perturbation theory, 10 pts)
   At $t = -\infty$ a particle is in its ground state in an 1D harmonic oscillator potential. At $t = 0$, a perturbation $V(x,t) = V_0\hat{x}e^{-t/\tau}$ is turned on. Calculate the probability at $t = \infty$ that the system will have made a transition to its first, and the second excited states (within the first order approximation)

6. (Approximation Method, 10 pts)
   Consider a simple harmonic oscillator of mass m and oscillation frequency, ω, with a perturbation, $H' = \alpha x^4$. Using Gauusian variational wavefunction to determine the most possible ground state energy. (Hint: $\int_0^\infty x^{2n} e^{-\frac{x^2}{a^2}}\, dx = \sqrt{\pi}\frac{a^{2n+1}(2n-1)!!}{2^{n+1}}$)

Classical Electrodynamics (2021 March)

Please use SI unit system. If not, please indicate the system you use.

1. Explain the following terms. (8*5%)

(a) Perfect conductor and super conductor

(b) Plasma frequency and skin depth

(c) Liénard-Wiechert potentials

(d) Lorentz gauge and Coulomb gauge

(e) Group and phase velocities

(f) Synchrotron radiation

(g) Complex Poynting's theorem

(h) Explain why the sky is blue during the daytime and red at sunset.

2. Derive the Green theorem step-by-step.

(a) Green's first identity and second identity. (5%)

(b) Application of the Green theorem to electrostatic boundary-value problems with the Green function satisfying $\nabla'^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ with $G_D(\mathbf{x},\mathbf{x}') = 0$ for $\mathbf{x}'$ on *S*. (10%)

4. Two concentric conducting shells of inner and outer radii *a* and *b* (*b*>*a*), respectively. The inner shell is connected to a potential $V(a,\theta)$ (to be given), while the outer shell is grounded $V(b,\theta)=0$.

(a) If $V(a,\theta)=V_0$ (constant), find the potential at $r<a$, $a<r<b$, and $r>b$. (10%)

(b) If $V(a,\theta)=V_0\cos^2\theta$, find the potential everywhere between the shells ($a<r<b$). (10%)

[Hint: use Legendre polynomials $P_0(x)=1,\ P_1(x)=x,$ and $P_2(x)=(3x^2-1)/2$].

5. Starting from the Maxwell equations, derive the dispersion relation (i.e. the relation between the wave frequency ω and the propagation constant k) for a plane electromagnetic wave in an infinite and uniform medium of conductivity σ, electrical permittivity ε, and magnetic permeability μ. (15%)

6. Prove that under Lorentz transformation E^2-B^2 and $\mathbf{E}\cdot\mathbf{B}$ are invariant. (15%)

Qualifying Examination – Statistical Mechanics

Spring, 2021

1. 15 points (**Maximum entropy principle and statistical ensembles**) Hint: use the method of Lagrange multipliers to find the extrema of a function with constraints.
   (a) Consider a biased, six-sided die. It is twice likely to yield an even number than an odd number when thrown. Find the probability distribution $\{p_n\}$ which maximizes the entropy $S = -\sum_n p_n \ln(p_n)$ subject to the two constraints: (1) $\sum_n p_n = 1$ and (2) $p_2 + p_4 + p_6 = 2(p_1 + p_3 + p_5)$.
   (b) Show that if we maximize the entropy $S = -k_B \sum_n p_n \ln(p_n)$ subject to the three constraints: (1) $\sum_n p_n = 1$, (2) $\sum_n E_n p_n = \langle E \rangle$ for the averaged energy, and (3) $\sum_n N_n p_n = \langle N \rangle$ for the averaged particle number, then the probability distribution $\{p_n\}$ is described by the grand canonical ensemble. That is $p_n = \frac{1}{\Xi} e^{-\beta(E_n - \mu N_n)}$, where $\Xi = \sum_n e^{-\beta(E_n - \mu N_n)}$.

2. 20 points (**Spins and vacancies on a surface and the negative temperature**) Consider a surface with N sites and a collection of non-interacting spin-1/2 particles. For each site the energy $\epsilon = 0$ if there is a vacancy and $\epsilon = -W$ if there is a particle present, where $-W < 0$ is the binding energy.
   (a) Let $N_\pm$ be the number of particle with spin $\pm\frac{1}{2}$, $Q = N_+ + N_-$ be the number of spins, N_0 be the number of vacancies, $M = N_+ - N_-$ be the surface magnetization. Note that $N_+ + N_- + N_0 = Q + N_0 = N$. In the microcanonical ensemble, compute the entropy $S(Q, M)$. (Hint: Calculate the number of states available to the system and express N_+, N_-, N_0 in terms of Q and M.
   (b) Let $q = Q/N$ be the dimensionless particle density and $m = M/N$ be the dimensionless magnetization density. Assuming that we are in the thermodynamic limit, where N, Q, and M all tend to infinity, but with q and m finite. Find the temperature $T(q, m)$. (Hint: Use Stirling's formula $ln\,N! \approx N\,ln\,N - N$.)
   (c) Show explicitly that T can be negative for this system. What does negative T means? What physical degrees of freedom have been left out that would avoid this strange property?

3. 20 points (**Bose condensation**) Consider a 3D non-interacting Bose gas with the energy dispersion $\epsilon(\vec{k}) = A|\vec{k}|^{1/2}$.
   (a) Obtain an expression for the density $n(T, z)$ where $z = exp(\mu/k_B T)$ is the fugacity. Simplify your expression as best as you can. You may find it convenient to define

$$Li_\nu \equiv \frac{1}{\Gamma(\nu)}\int_0^\infty dt \frac{t^{\nu-1}}{z^{-1}e^t - 1} = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu}.$$

Note $Li_\nu(1) = \zeta(\nu)$, the Riemann zeta function.

(b) Find the critical temperature for Bose condensation $T_c(n)$. Your expression should only include the density n, the constant A, physical constants, and numerical factors (which may be expressed in terms of integrals or infinite sums.)

(c) What is the condensate density n when $T = \frac{1}{2}T_c$.

4. 20 points (**Ideal Fermi gas**) Consider N non-interacting fermions of mass m in 3D volume V.

(a) Find the Fermi energy, E_F.

(b) Find the degenerate pressure at $T = 0$.

(c) Find the specific heat, C_V at $T \ll E_F$.

5. 25 points (**Ising model and the mean field theory**) Consider the Ising model Hamiltonian

$$H = -J\sum_{\langle i,j\rangle} \sigma_i\sigma_j - H_{ext}\sum_i \sigma_i,$$

where $\sigma_i = \pm 1$ is the spin at site i, $\langle i,j\rangle$ denotes nearest neighbor sites, and H_{ext} is the external magnetic field. Denote z the lattice coordination number, i.e., number of nearest neighbors for each site, and N the number of lattice sites.

(a) Derive the mean field Hamiltonian H_{MF} by decomposing σ_i into its average and a fluctuation term, and assume that the quadratic terms in the fluctuations can be neglected. We will also assume that the dimensionless local magnetization $\langle\sigma_i\rangle = m$ is independent of position i.

(b) From H_{MF}, show that the Boltzmann weights are completely determined by a Hamiltonian of noninteracting spins in an effect mean field H_{eff}. Derive the self-consistent equation for the magnetization m.

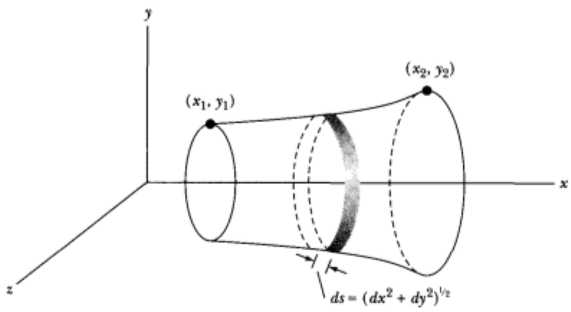
(c) Derive the dimensionless free energy $f(m,h,\theta) \equiv F/(NzJ)$ where F is the free energy, $\theta \equiv k_B T/(zJ), h \equiv \mathrm{H_{ext}(zJ)}$ are dimensionless variables.

(d) Show that $\partial f/\partial m = 0$ gives the same equation for magnetization.

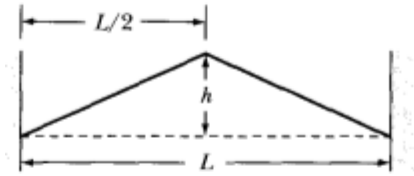
(e) Assume $h = 0$, expand f in power of m up to forth order and determine the critical θ_c at which the sign of quadratic term changes.

(f) Assume $h = 0$, show that for $\theta > \theta_c$, $f(m,\theta)$ has a single minimum at $m = 0$. When $\theta > \theta_c$, there are three local minima $m_+, 0, m_- = -m_+$. Find $m_\pm$ from the expansion of f you obtain in (e).

1. (Calculus of variations, 20 points) Consider the surface generated by revolving a line connecting (x_1, y_1) and (x_2, y_2) about an axis coplanar with these two points. Find the equation of the line connecting the points such that the surface area generated by the revolution is a minimum.



2. (Damped waves, 20 points) A guitar string of length L, mass per unit length ρ, and damping constant α is under a tension of T. Find the displacement $q(x,t)$ if the string is plucked in the middle by a height h at $t = 0$ and released from rest.

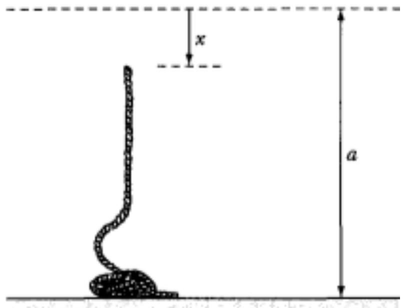


Hint: The wave equation resembles $\frac{d^2q}{dt^2} = \frac{T}{\rho}\frac{d^2q}{dx^2} - \alpha\frac{dq}{dt}$.

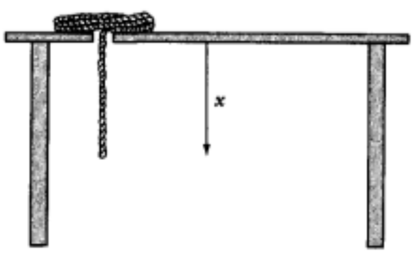
3. (Rocket motion, 10 points) A rocket leaves Earth's surface in a vertical direction. The exhaust velocity is u, and the constant fuel burn rate is α. Let the initial mass be m_0 and the mass at fuel burnout be m_f. Calculate the altitude and speed of the rocket at fuel burnout.

4. (Rope problem, 20 points)

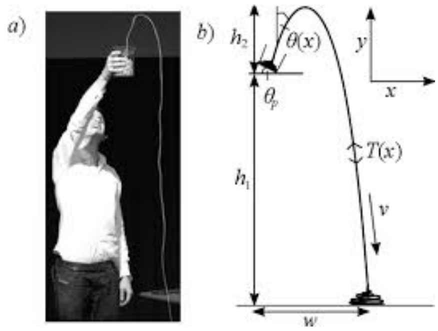
(a) A rope of mass per unit length ρ and length a suspended just above a table. If it is released from rest at the top, find the force on the table when a length x of the rope has dropped on the table.



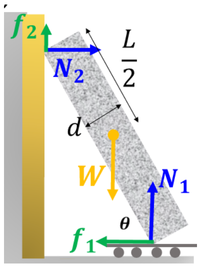
(b) A smooth rope is places above a hole in a table. One end of the rope falls through the hole at $t = 0$, pulling steadily on the remainder of the rope. Find the velocity and acceleration of the rope as a function of the distance to the end of the rope x. Ignore all friction.



5. (Chain fountain, 10 points) Place a chain of beads inside a jar. When one end of the chain is yanked from the jar and is allowed to fall to the floor beneath, a self-sustained flow of the chain will rise up from the jar and form an arch that ascends into the air. Find the shape of this chain and how the speed, v, depends on h_1.



6. (Statically indeterminate, 5+5+10 points) The Ladder-Wall problem is an illustrative example for students learning statics. Use L and W to denote the length and weight of the ladder with elevation angle θ. Neglect the width d.



(a) Write down the conditions that link the four friction and normal forces, $f_{1,2}$ and $N_{1,2}$, if the ladder is at static equilibrium.

(b) Assume Mother Nature prefers the wall to minimize its "squeeze", $N_2\cos\theta$-$f_2\sin\theta$, on the ladder to minimize the distortion energy. Set $N_2\cos\theta$-$f_2\sin\theta = 0$ and combine with the conditions in (a) to solve for N_1.

(c) Note that $f_{1,2}$ should never exceed $\mu_{1,2}N_{1,2}$ where $\mu_{1,2}$ denotes the static friction coefficient from floor and wall. Find out at what range of θ will the answer in (b) contradicts this condition. When this occurs, $N_2\cos\theta$-$f_2\sin\theta$ will have to adopt the lowest possible positive value other than 0. How will you tackle this case and what is the new solution for N_1?