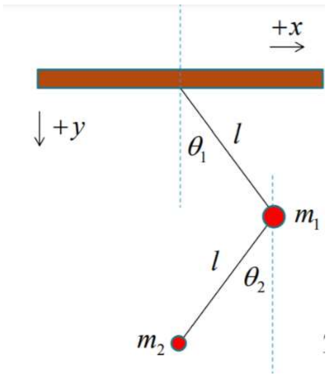
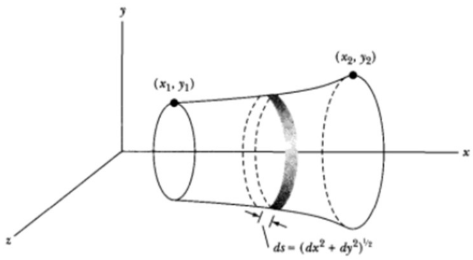
Qualification Exam. Problem Set
Classical Mechanics

Spring, 2022

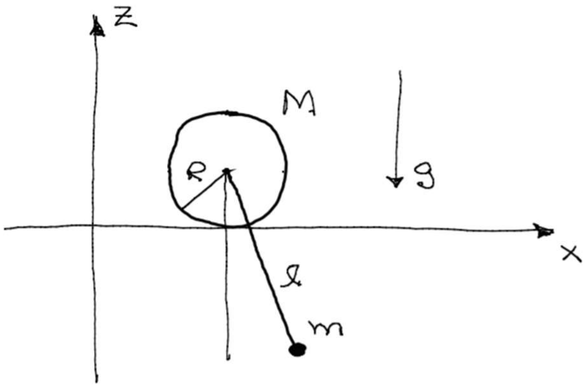
1. A double pendulum of equal lengths, unequal masses is shown as in Fig.1.
   (a)、(10 points) Find the Hamiltonian and Hamilton's equations of motion.
   (b)、(10 points) Assuming both $\theta_1, \theta_2 \ll 1$, find the periods of this harmonic motion.



2. (15 points) Consider the surface generated by revolving a line connecting points (x_1, y_1) and (x_2, y_2) about an axis coplanar with these two points. Find the equation of the line connecting the points such that the surface area generated by the revolution is a minimum.



3. A hollow cylinder of mass M and radius R rolls without slipping on a table. The end of the cylinder hangs off the edge of the table and is constructed in such a way that a pendulum is suspended from the axis of the cylinder. The pendulum rod, assumed rigid and massless, has length l, and the pendulum bob has mass m. Assume that the pendulum can rotate without friction in its support.



   (a)、(10 points) Define carefully a set of independent generalized coordinates for this system, and find the Lagrangian L.
   (b)、(5 points) Write down the Lagrange's equation and find two conserved quantities for this system.

4. A particle of mass m described by one generalized coordinate q moves under the influence of a potential $V(q)$ and a damping force proportional to its velocity as $-2m\gamma q$.

(a)、(5 points) Show that the following Lagrangian $L = e^{2\gamma t}\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)$ gives the desired equation of motion.

(b)、(5 points) Obtain the Hamiltonian $H(q, p, t)$ for this system.

(c)、(5 points) Consider the following generating generating function

$$F = e^{\gamma t} qP - QP$$

Obtain the canonical transformation from (q, p) to (Q, P) and the transformed Hamiltonian $K(Q, P, t)$.

(d)、(5 points) Pick $V(q) = \frac{1}{2}m\omega^2 q^2$ as a harmonic potential with a natural frequency ω. Show that the transformed Hamiltonian yields a constant of motion.

(e)、(Bonus 10 points) Obtain the solution $Q(t)$ for the damped oscillator in the under damped case $\gamma < \omega$ by solving Hamilton's equations in the transformed coordinates. Then, write down the solution $q(t)$ using the canonical coordinates obtained in part (c).

5. A rocket of mass m is fired horizontally from height h and eventually returns to Earth.

(a)、(10 points) Find the possible trajectories.

(b)、(5 points) What are the corresponding conditions for these orbits?

6. (15 points) Solve the problem of the motion of a point projectile in a vertical plane, using the Hamilton-Jacobi method. Find both the equation of the trajectory and the dependence of the coordinates on time, assuming the projectile is fired off at time $t = 0$ from the origin with the velocity v_0, making an angle α with the horizontal.

Classical Electrodynamics (2022 March)

Please use SI unit system. If not, please indicate the system you use.

1. Explain the following terms. (6*5%)

(a) Plasma frequency and skin depth

(b) Lorentz gauge and Coulomb gauge

(c) Liénard-Wiechert potentials

(d) Poynting's theorem

(e) Group and phase velocities

(f) Synchrotron radiation

2. A plane electromagnetic wave with vector field E_i and B_i is normally incident from the air upon a perfectly conducting flat surface. The reflected wave has vector fields E_r and B_r.

(a)What is the relation between E_r and E_i on the surface? State the reason. (10%)

(b)What is the relation between B_r and B_i on the surface? State the reason. (10%)

3. Derive the Green theorem step-by-step.

(a) Green's first identity and second identity. (10%)

(b) Application of the Green theorem to electrostatic boundary-value problems with the Green function satisfying $\nabla'^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ with $G_D(\mathbf{x},\mathbf{x}')=0$ for $\mathbf{x}'$ on *S*. (10%)

4. Write down the relativistic equation of motion (in vector form) of a charged particle (with charge *q* and rest mass *m*) in the presence of electric field $\mathbf{E}$ and magnetic induction $\mathbf{B}$.(10%)

5.

(a)Starting from the Maxwell equations, derive the dispersion relation (i.e. the relation between the wave frequency ω and the propagation constant k) for a plane electromagnetic wave in an infinite and uniform medium of conductivity σ, electrical permittivity ε, and magnetic permeability μ. (10%)

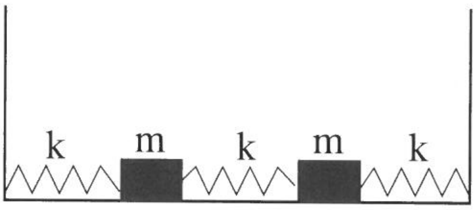
(b)Assume that the medium is a good conductor, derive an expression for its skin depth λ. (10%) [vector formula: $\nabla\times(\nabla\times A)=\nabla(\nabla\cdot A)-\nabla^2 A$]

Quantum Mechanics Qualification
Spring, 2022

Problem 1 Answer the following questions *briefly*
(a) 6% What are generators of rotations?
(b) 6% Find the Hermitian conjugate of $x^4 \frac{d}{dx}$ and $3^{i|1\rangle\langle 2|}$.
(c) 4% What kind of particles does the Dirac equation describe?
(d) 8% Explain the following terms briefly: (a) entangled states (b) Clebsch-Gordan coefficients.
(e) 6% Consider a particle whose position is denoted by $\vec{r} = (x, y, z)$ and total angular momentum is denoted by $\vec{J}$. Find $\exp(iJ_z\phi/\hbar)\, x^2 \exp(-iJ_z\phi/\hbar)$ in terms of x and ϕ.

Problem 2 Consider two particles of the same mass m in a one-dimensional box. These particles are connected by a spring with spring constant k and are connected to the wall by another two springs with the same spring constants k as shown in Fig.1. Classically, when all particles are at rest, all springs are unstreched with the same length.
(a) 10% Suppose that two particles are non-identical. Find all energy eigenvalues and the normalized ground state wavefunction.
(b) 10% Suppose that two particles are identical. Find the energy eigenvalues of this system and the minimum relative distance $\sqrt{\langle (x_1 - x_2)^2 \rangle}$ between two particles in the eigenstate for the following two cases: (1) particles are fermions, (2) particles are bosons.

Fig. 1



Problem 3
Given an arbitrary wavefunction $\psi(x)$ at $t = 0$, the propagator $U(x, t; x', 0)$ can lead us to its form at any later time t under the influence of potential $V(x)$:

$$\psi(x,t) = \int_{-\infty}^{\infty} U(x, t; , x', 0)\psi(x')dx'$$

(a) 7% Find the $U(x, t; x', 0)$ for a free particle of mass m.
(b) 5% Following (a), in the presence of the general potential $V(x)$, $U(x, t; , x', 0)$ can be evaluated by integrations using the path integral formulation. What is its general form (specify the integration and integrand) in the path integral formation?
(c) 8% By setting $x = x'$ and integrating $U(x, t; , x, 0)$ over x, one obtains

$$\int_{-\infty}^{\infty} U(x, t; , x, 0)dx = \sum_n e^{-iE_n t} = \int_0^{\infty} e^{-iEt} D(E)dE,$$

where E_n are energy eigenvalues and $D(E) = \sum_n \delta(E - E_n)$ is the density of states. Now, it is known that for the harmonic oscillator, the propagator is given by

$$U(x, t; , x', 0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega t}} \exp\left\{ \frac{im\omega}{2\hbar \sin \omega t} \left[(x^2 + x'^2)\cos \omega t - 2xx' \right] \right\},$$

where ω is the angular frequency of the oscillation. Derive the energy eigenvalues from the above propagator.

Problem 4
(a) 4% Find $\mathbf{L} \cdot \mathbf{S}\, Y_1^{+1}(\theta, \phi)|+\rangle$ in terms of $Y_l^m(\theta, \phi)$, $|+\rangle$, and $|-\rangle$, where $\mathbf{L}$ is the orbital angular momentum vector

operator, $\mathbf{S}$ is the spin vector operator and $\{|+\rangle, |-\rangle\}$ are eigenkets to S_z.
(b) 6% Consider a system of two particles with spins $s_1 = 1$ and $s_2 = 1/2$. By considering the addition of two spins, find all possible eigenvalues to the operator $(2\mathbf{S}_1 - 3\mathbf{S}_2)^2$.

Problem 5 10%
Consider a central scattering potential $V(r) = V_0$ for $r < a$ and $V(r) = 0$ elsewhere, where V_0 is a constant. Use the Born approximation to evaluate the total scattering cross section in the limit of $(ka \ll 1)$, where k is the momentum of the incident particle.

Problem 6 10%
Consider a one-dimensional harmonic oscillator with mass m and natural frequency ω. The oscillator is in its ground state $|0\rangle$ at $t = -\infty$. A time dependent potential is switched on as follows

$$V(x,t) = \frac{V_0 x}{1 + (t/\tau)^2}, \tag{1}$$

where τ is a positive constant and V_0 is a constant as well. Find the probability that the oscillator stays in the same state $|0\rangle$ at $t = \infty$ to order V_0^2.

Qualifying Examination – Statistical Mechanics

2022, spring

Please explain the logic behind your answers.

Problem 1. *Ising model(20 points):* Considering the Ising model $H = -J\sum_{\langle i,j\rangle} \sigma_i\sigma_j$ where $\sigma_i = \pm 1$ and $\langle i, j\rangle$ denotes the nearest-neighbor pairs of sites.

1. For a system of coordinate number z, use mean-field approximation to find the critical temperature T_c below which spontaneous magnetization exists.
2. Show that the magnetic susceptibility $\chi \propto (T - T_c)^{-1}$ at $T \gg T_c$
3. Use entropy argument to show that in 1D there is no phase transition, i.e., $T_c = 0$.

Problem 2. *A simplified model of hemoglobin(10 points):* Hemoglobin is a protein in the blood that carries oxygen from the lungs to the muscles. It is formed by four units. Each unit can carry an O_2 molecule or not. For a hemoglobin, there will be 2^4 configurations as shown in Fig.1. (Each unit has two choices, carries nothing or an O_2 molecule. Therefore we have 2^4 possible choices.) As described in the caption of Fig.1, we can model the hemoglobin by assigning the binding energy $-\varepsilon_0$ and the additional gain, $-J$, if two nearby unit both bind with O_2(We try to pack O_2 to our hemoglobin as much as possible, a simple minded physicist guess). For example, considering the

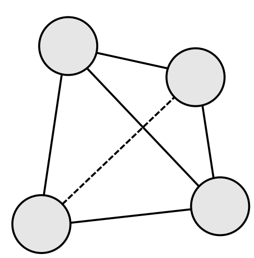


Figure 1: Schematic model of hemoglobin. The gray circles represent the unit that can carry an O_2 molecule. We assume the four units form a tetrahedral network structure. We can approximate the hemoglobin using a simplified model: The energy gain for an O_2 to bind with a unit is $-\varepsilon_0$. If two nearby units both contain O_2, the energy can further reduced by energy $-J$.

configuration in Fig. 2, the total energy of the simplified model is $(-\varepsilon_0) \times 2 + (-J) \times 1$. Consider we have $N/4$ hemoglobin, labeled by $\alpha = 1 \sim \frac{N}{4}$. On each hemoglobin, we have 4 units which can bind with O_2 molecules. We can use $\tau_{\alpha,i} = 0, 1$ for $i = 1 \sim 4$ to denote whether the i-th unit on the α-th hemoglobin is occupied by an O_2 molecule or not.

1. Write down the expression of the grand canonical partition function of the system with $\frac{N}{4}$ hemoglobin. Express the result using $x = exp\left[\frac{\varepsilon_0+\mu}{k_BT}\right]$ and $y = exp\left[\frac{J}{k_BT}\right]$. (Hint: we

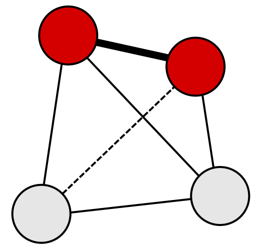


Figure 2: One particular configuration of hemoglobin. The red circles represent the units that are occupied by O_2 molecules. The thick line represent the additional energy reduction, $-J$, for nearby units are occupied by O_2 molecules. There will be other 5 configurations with equal energy.

consider the molecules of hemoglobin to be independent. So, we can factorize the partition function of the system into products of partition functions of individual hemoglobin molecule. Then, within each Hemoglobin, we can enumerate all the possible configurations and the corresponding energy to construct our partition function. Be careful about the number of configurations with equal energy. The degeneracy of energy in this system is not very regular, so one needs to specify them explicitly.)

2. Evaluate the average number $\langle M \rangle = \sum_{\alpha=1}^{N/4} \sum_{i=1}^{4} \langle \tau_{\alpha,i} \rangle$ of adsorbed molecules as a function of x, y.

Problem 3. *3D Ideal fermi gas (20 points):*

1. Consider a 3D ideal Fermi gas of spin $S = \frac{5}{2}$ and mass m at zero temperature in a box with volume L^3, calculate the total internal energy by summing the electron energy all the way up to the fermi energy. Express the answer using m, L and $\hbar$.

2. Calculate the fermi pressure of this system.

Problem 4. *Spins and vacancies on a surface and the negative temperature(20 points):* Consider a surface with N sites and a collection of non-interacting spin $\frac{1}{2}$ particles. For each site the energy $\varepsilon = 0$ if there is a vacancy and $\varepsilon = -W$ if there is a particle present, where $-W < 0$ is the binding energy.

1. Let $N_\pm$ be the number of particle with spin $\pm\frac{1}{2}$, $Q = N_+ + N_-$ be the number of spins, N_0 be the number of vacancies, $M = N_+ - N_-$ be the surface magnetization. We have $N_+ + N_- + N_0 = N = Q + N_0$. In the microcanonical ensemble, compute the entropy $S(Q, M)$. (Hint: Calculate the number of states available to the system and express N_+, N_-, N_0 with Q and M.

2. Let $q = \frac{Q}{N}$ be the dimensionless particle density and $m = \frac{M}{N}$ be the dimensionless magnetization density. Assuming that we are at the thermodynamic limit where N, Q and M all tend to infinity, but with q and m finite. Find the temperature $T(q, m)$. (Hint: Use Stirling's formula $\ln N! \approx N \ln N - N$.)

3. Show explicitly that T can be negative for this system. What does negative T mean? What physical degrees of freedom have been left out that would avoid this strange property?

Problem 5. *Bosons in Harmonic traps(30 points):* Let us consider a particle in the anisotropic harmonic-oscillator potential $V(\boldsymbol{r}) = \frac{1}{2}\left(K_x x^2 + K_y y^2 + K_z z^2\right)$. Therefore, we can consider the system as three independent harmonic oscillators in three different directions x, y and z. The corresponding energy levels can be parameterized by three non negative integers (n_x, n_y, n_z) as

$$E(n_x, n_y, n_z) = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z. \tag{1}$$

1. Let us consider n_i are continuous variables and neglect the zero-point energies (The $\frac{1}{2}\hbar\omega_i$ in $E(n_x, n_y, n_z)$.), we can simplify the energy as $E \approx \varepsilon_x + \varepsilon_y + \varepsilon_z$ where $\varepsilon_i = n_i\hbar\omega_i$. If we define $G(E)$ as the number of states below energy a specific energy E. Derive the expression of $G(E)$ in terms of E, $\omega_x, \omega_y, \omega_z$ and $\hbar$. (Hint: There are several way to derive this result. One approach is to consider the problem in the (n_x, n_y, n_z) space. What is the geometric meaning of the states below energy E?)

2. Another way to understand density of state is $g(E) = \frac{dG(E)}{dE}$. Use the above expression to get the density of states.

3. We can evaluate the particles in the excited states using

$$N_{ex} = \int_0^\infty dE g(E) \frac{1}{e^{E/k_B T} - 1}. \tag{2}$$

   Here, we assume N_{ex} reaches its maximum for $\mu = 0$. The definition of the transition temperature, T_c, of Bose-Einstein condensation is when the total number of particles can be just accommodated in excited states. That is,

$$N = N_{ex}(T = T_c, \mu = 0) = \int_0^\infty dE g(E) \frac{1}{e^{E/k_B T} - 1}. \tag{3}$$

   $\int_0^\infty dx \frac{x^{\alpha-1}}{e^x - 1} = \Gamma(\alpha)\zeta(\alpha)$. Here, $\zeta(\alpha) = \sum_{n=1}^\infty n^{-\alpha}$ is the Riemann zeta function and $\Gamma(\alpha)$ is the gamma function. Derive the transition temperature of this system and express the result using $N, \omega_x, \omega_y, \omega_z$ and the Riemann zeta function.