Qualification Exam for Ph.D Candidacy

Statistical Mechanics

1. Please explain the following concepts clearly (5 points each)

(a) The fundamental assumption of the equilibrium statistical mechanics
(b) Bose-Einstein condensation
(c) Fermi pressure
(d) First and second order phase transition
(e) Ginsburg Landau free energy
(f) Canonical ensemble and grand canonical ensemble
(g) Pauli paramagnetism

2. Consider a collection of noninteracting cold polar molecules. In very low temperature, a single molecule can be treated as a quantum rigid rotor, so that the system Hamiltonian is given by

$$\mathrm{H} = \sum_{\mathrm{i}}^{\mathrm{N}} \left(\frac{\mathbf{p}_{\mathrm{i}}^{2}}{2\mathrm{m}} + \mathrm{B}\mathbf{J}_{\mathrm{i}}^{2} \right)$$

wherem is molecule mass, p is momentum operator, and J is angular momentum operator of the molecule. B is a rotational constant.
(a) (5 pts) Please write down the eigen-energy of a single particle. Explain the definition of your notations, especially those good quantum numbers.
(b) (10 pts) Assuming these polar molecules are classical particle, write down the general expression of the partition function at temperature T.
(c) (5 pts) Using the above expression to calculate the free energy and entropy

3. Consider a system of *N* distinguishable particles, which have two energy levels, $E_0 = -\mu\mathrm{B}$ and $E_1 = \mu B$, for each particles. Here μ is magnetic moment and *B* is magnetic field. The particles populate the energy levels according to the classical distribution law.
(a) (10 pts) Calculate the average energy of such system at temperature *T*, and
(b) (5 pts) the specific heat of the system.
(c) (5 pts) Calculate the magnetic susceptibility.

4. Consider a 3D system of *N* noninteracting particles, whose energy dispersion is *E* =

$c|\mathbf{p}|$, where c is speed of light. Assuming they are classical particles at an equilibrium temperature T. The system volume is V. Use the following steps to construct its thermodynamical properties:

(a) (10 pts) Calculate the partition function of such system, Q, and then obtain its Helmholtz free energy, F.

(b) (5 pts) Use Q to show that the average energy, $U = 3N\mathrm{k_B}T$.

(c) (5 pts)Use F to show the pressure, $P = Nk_BT/V$. (5%)

(d) (5 pts) Use F to calculate entropy S. (5%)

Qualifying examination on Classical Electrodynamics (2/14/2011)

1. (Chap. 1, Boundary-value problems in electrostatics，20 分)

   A point charge q is placed at $(0,0,d)$ inside an <u>un</u>grounded hollow metal sphere centered at the origin and of inner/outer radius of a/b and $d < a$. Use the **image method** to determine the charge distribution on both the inner and outer surfaces.

2. (Chap. 4, Multipoles, electrostatics of macroscopic media, dielectrics，25 分)

   A charge-neutral dielectric sphere of radius a with dielectric constant ε is placed in an initially uniform electric field which at large distances from the sphere is directed along the z axis and has magnitude E_0. Please

   (a) Solve for the potential both inside and outside of the sphere.

   (b) Find the polarization charge density on the surface of the sphere.

3. (Chap. 7, Plane electromagnetic waves and wave propagation，30 分)

   Copper plates were used as mirrors in ancient times tell us that electromagnetic (EM) waves in the optical range are mostly reflected. We also know that x-ray can be utilized to detect possible dislocations in the fuselage of airplanes. So there must be a characteristic frequency ω which separates these two behaviors.

   (a) Within the Drude classical model, the metallic electrons obey the equation of motion: $m\vec{a} = q\left[\vec{E} + \left(\vec{v} \times \vec{B}/c\right)\right] - m\vec{v}/\tau$ where τ is the mean free time. Since the drift velocity v is much smaller than the speed of light c, you can neglect the force from $\vec{B}$ field. Please derive the expression for **AC conductivity** $\sigma(\omega) \equiv j/E$ where j denotes the current density.

   (b) Derive the AC dielectric constant $\varepsilon(\omega)$ for good conductors $\omega\tau >> 1$. Identify the **plasma frequency** ω_P, which renders ε negative when $\omega < \omega_P$.

   (c) Explain why a negative ε implies dissipations for the EM waves. Determine the **skin depth**, at which the amplitude is diminished by $1/e$ into the metal.

   (d) What are the **Kramers-Kronig relations**? Check if the $\sigma(\omega)$ and $\varepsilon(\omega)$ you obtained in (a) and (b) satisfy them. If not, explain why. Be careful not to assume $\omega\tau >> 1$ since the relations require you to integrate over all ω.

4. (Chap.8, Wave guides and resonant cavities，25 分)

   A coaxial transmission line consists of a long straight wire of radius a and a cylindrical conducting sheath of radius b. The electric and magnetic fields are given by: $\vec{E}(\rho,\phi,z,t) = \dfrac{E_0 \cos(kz - \omega t)}{\rho}\hat{e}_\rho$ and $\vec{B}(\rho,\phi,z,t) = \dfrac{B_0 \cos(kz - \omega t)}{\rho}\hat{e}_\phi$.

   (a) What is the relation between E_0 and B_0?

   (b) Find the line charge density and current on the inner conductor.

   (c) Determine the time-averaged power flow along the line..

Qualification Examiniation on Physics A
For Ph. D. Candidates

3/5/2011

Classical Mechanics (5 problems, 2 pages)

Problem [1] (15%)

Consider a particle of mass m moves in a circular orbit under the influence of a central potential $V(r) = -km/r^n$, where k is a positive constant. For small perturbation to a circular orbit, what is the range of n for a stable orbit?

Problem [2] (20%)

Consider a double pendulum consists of two particles of mass m_1 and m_2 suspended by masless rods as shown in Fig. 1. l_1, l_2 are the lengthes of the rods.

- [1] Assuming that all motion is in a vertical plane, find the Lagrangian and the equations of motion.
- [2] Find the normal-mode frequencies assuming small angles and $m_1 = m_2$, $l_1 = l_2$.

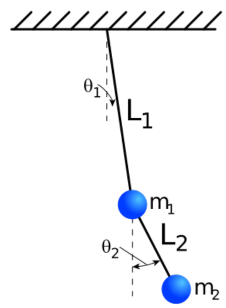

Figure 1:

Problem [3] (20%)

A uniform rod with mass M and radisu ρ rolls without slipping in a bigger cylinder of radisu R as shown in Fig 2.

- [1] Find the Lagrangian, the equation of constraint, and equations of motion.
- [2] Calculate the frequency of small oscillation.

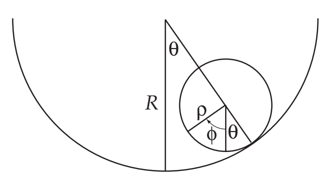

Figure 2:

Problem [4] (15%)

Calculate the minimum total energy of a proton (in the lab system) which hits another proton at rest (in the lab system) to produce an anti-proton in the following process

$$p + p \rightarrow p + p + p + \bar{p}.$$

Note: The rest energy of proton and anti-proton are 938 Mev but you may use 1 Gev in your calculation. You must use the relativistic forms of momentum and energy.

Problem [5] (30%)

Consider a simple harmonic oscillator with

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right).$$

- [1] Apply the canonical transformation with generating function $F = F_1(q, \mathbb{Q}) = \frac{1}{2}m\omega q^2 \cot \mathbb{Q}$ to find the transformed Hamiltonian K. Find the solution for $\mathbb{P}(t)$, $\mathbb{Q}(t)$, and then $q(t)$.
- [2] Apply the Hamiltonian-Jacobi theory to find $q(t)$, assuming the initial conditions $q(0) = q_0$ and $p(0) = p_0$. (Hint: The generating function $F = F_2(q, \mathbb{P}) - \mathbb{Q}\mathbb{P}$ and the transformed Hamiltonian $K = 0$).
- [3] Use action-anle variable to find the frequency of oscillation, starting from defining the constant action variable $J = \oint p dq$.

You might need to use following integral:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right),\ a < 0, b^2 > 4ac.$$

Quantum Mechanics Qualification, March 6, 2011.

You must provide the details or reasonings to justify your answers.

1. (10+5+5%) **Three noninteracting electrons** are put in an 1-D infinite potential well, $V(x) = 0$ for $0 \le x \le L$ and $V(x) = \infty$ elsewhere. (1) Write down the spatial and spinal wave function of one of the ground states. (2) What is the total angular momentum of this state? (3)What is the energy and the degeneracy of the ground state?

2. (5+10%) The symbol $|j, m\rangle$ denotes the simultaneous eigenstate of $\hat{J}^2$ and $\hat{J}_z$ with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$ respectively. Given a prepared state

$$|\psi\rangle = \alpha|j, m+1\rangle + \beta|j, m-1\rangle$$

where α and β are complex numbers and $|\alpha|^2 + |\beta|^2 = 1$, calculate $\langle \hat{J}_x \rangle$ and $\langle \hat{J}_y^2 \rangle$.

3. (10+10%) The Hamiltonian of a system is given by

$$H = \frac{17}{8} a^\dagger a - \frac{15}{16}\left(a^2 + (a^\dagger)^2\right) + \frac{41}{16}$$

where

$$a = \frac{1}{\sqrt{2}}(q + ip)\,,\ a^\dagger = \frac{1}{\sqrt{2}}(q - ip)\,,\ \text{and } [q, p] = i.$$

(1) What are the eigen-energies of this system? (2) What is the ground state wave function? Remember to fix the normalization.

4. (10+10%) Consider a central scattering potential $V(r) = V_0/r$ for $r < a$ and $V(r) = 0$ elsewhere. Where V_0 is a constant. (1) Calculate the Born approximation scattering amplitude. (2) Use the result to evaluate the total scattering cross section in the limit of very low incident energy ($ka \ll 1$).

5. (10+5%) A system is governed by the Hamiltonian H_0. Denote the eigenstates by $|n\rangle$ with energy E_n, $n = 1, 2, 3, \cdots$, i.e., $H_0|n\rangle = E_n|n\rangle$. We assume there is no degeneracy. At $t = 0$, a potential $V(t)$ is turned on whose matrix elements $\langle n|V(t)|m\rangle$ vanish except that $\langle 2|V(t)|3\rangle = V_0 e^{+i\omega t}$ and $\langle 3|V(t)|2\rangle = V_0 e^{-i\omega t}$, where V_0 is real.
(1) Assume that at $t = 0$ the system is at state $|2\rangle$, using time-dependent perturbation theory to first order in V_0, evaluate the probability of finding the state in $|2\rangle$ at time $t > 0$. (2)Discuss the physics when the perturbation is in resonant ($\omega = E_3 - E_2$).

6. (3+7%) (1)Explain the physics of hyperfine splitting. (2)Given that the hyperfine splitting in the hydrogen atom produces 21 cm radiation, find the corresponding wavelength (in cm) for the deuterium atom (consists of one proton and one neutron) with the nucleus spin-1, $g_p = 2 \times (2.79)$, and $g_n = 2 \times (-1.91)$.