2010 Fall

2010 Qualify Examination: Classical Mechanics

1. A particle of mass m can slide freely along a wire AB whose perpendicular distance to the origin O is h (see Fig.1). The line OC rotates about the origin at a constant angular velocity $\dot{\theta} = \omega$. The position of the particle can be described by θ and the distance q to the point C. If the particle is subject to a gravitational force, and if the initial conditions are $\theta(0) = 0$, $q(0) = 0$, and $\dot{q}(0) = 0$, answer the following questions.

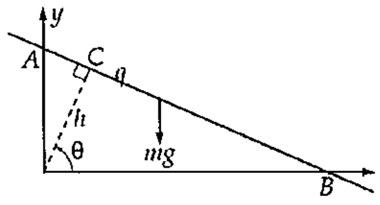


FIG. 1: Setup of problem 1

(a) 10% Find the time dependence of q and sketch $q(t)$ versus t.
(b) 5% If we take $y = 0$ as the zero point for the potential energy, find the total energy at time t.
(c) 5% Find the Hamiltonian for describing the particle at time t.

2. **15%** Apply the Hamilton-Jacobi theory to find $x(t)$ for the simply harmonic oscillator with $H = \frac{p^2}{2m} + \frac{1}{2}kx^2$ with the initial conditions: $x(0) = x_0$ and $p(0) = p_0$.

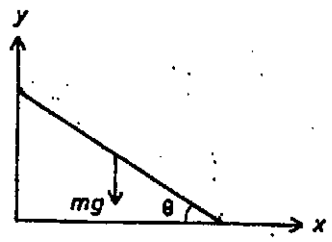


FIG. 2: Setup of problem 3

3. A uniform stick of length $2a$ is held temporarily so that one end leans against a frictionless vertical wall and the other end rests on a frictionless floor making an angle $\theta = \theta_0$ with the floor. When the stick is released, it will slide down under the influence of gravity (see Fig. ??).
(i) 7% Find the expression (as an integral) for the time that it will take for the stick to reach a new angle θ.
(ii) 8% Find the angle when the upper end of the stick leaves the wall.

4. The dynamics for a particle of mass m can be described the Lagrangian

$$L = \frac{1}{2}mv^2 - q\phi(\vec{r}) + q\vec{A} \cdot \vec{v}, \tag{1}$$

where $\vec{r}$ is the displacement vector of the particle, $\vec{v} = \dot{\vec{r}}$, q is a constant, and $\vec{A} = \vec{B} \times \vec{r}/2$ with $\vec{B}$ being a constant vector.

(a) 8% Find the force that acts on the particle. From the form of force, identify a physical system that the Lagrangian may describe.

(b) 10% Find the generalized momentum $\vec{p}$, the Hamiltonian and the total energy for the corresponding physical system.

(c) 5% Suppose that $\vec{B} = B\hat{z}$, $\phi(\vec{r}) = \phi(r)$ (i.e., $q\phi$ is a central-potential), and the motion of the particle is confined to the $x - y$ plane. Find the constant(s) of the motion in the polar coordinates, r and θ, in addition to the total energy.

5. Consider two oscillators with the same mass m and spring constant k. If they are coupled by the potential $V = \epsilon x_1 x_2$, where ϵ is a constant and x_i are coordinates of the oscillators.

(a) 5% Find the condition that two oscillators may not be oscillating.

(b) 4% By choose appropriate unit, we may set $m = 1$ and $k = 3/2$. Suppose $\epsilon = 1/2$. Find the angular frequencies of oscillation.

(c) 10% Following (b), if at $t = 0$, we have $x_1 = 1, x_2 = 0, \dot{x}_1 = 0$, and $\dot{x}_2 = 0$. Find $x_1(t)$ and $x_2(t)$.

6. 8% Consider the pion photoproduction reaction

$$\gamma + p \rightarrow p + \pi_0,$$

where p is the proton, π_0 is the neutral pion, the rest energy is 938 MeV for the proton and 135 MeV for the neutral pion. If the initial proton is at rest in the laboratory, find the minimum energy of the gamma-ray γ such that the reaction can happen in the laboratory.

Qualification Exam for Ph.D. Candidacy

Statistical Mechanics
(September 8, 2010)

Spring 2010, Keh-Ying Lin

1. What is the fundamental assumption of the equilibrium statistical mechanics?

2. A crystal consists of N identical atoms. At very low temperature, the interaction between atoms can be neglected. If the ground state of each atom is doubly degenerate, what is the entropy of the crystal at zero absolute temperature?

3. Consider a classical ideal gas at temperature T. The energy of a molecule is

$$\epsilon = ax^2 + bx + \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) \tag{1}$$

where a and b are constants. Determine the average energy of a molecule.

4. The Hamiltonian of the one-dimensional Ising model is

$$H = -J \sum_{j=1}^{N-1} \sigma_j \sigma_{j+1} \tag{2}$$

where J is a positive constant, and $\sigma = \pm 1$ (two spin states). Calculate the partition function of this model.

5. Two identical bodies have the same constant heat capacity C. Their initial temperatures are T_1 and T_2 ($T_1 < T_2$). These bodies are connected to a reversible heat engine, which absorbs heat from the hot body and releases heat to the cold body until both bodies approach to the same final temperature T. Calculate T. You should write down the procedure of your derivation.

6. A surface with N_0 absorbing centers has $N(< N_0)$ gas molecules absorbed on it. The partition function of a single molecule is $a(T)$, and the interaction between the absorbed molecules can be neglected. Find the chemical potential of the absorbed molecules by constructing the grand partition function.

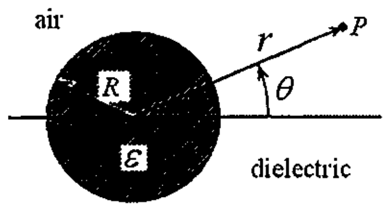
Ph.D. Qualify Exam: **Classical Electrodynamics**

September 2010

1. (10%, 10%) A conducting metal sphere has radius R and charge Q. Half of the sphere is embedded in a dielectric material with permittivity ε and half of the sphere is in the air (ε_0).

(a) Find the electric field at point P outside the sphere in the air. [Hint: use the spherical coordinate with azimuthal symmetry $\mathbf{E}(r,\theta)$.]

(b) Find the charge density on the upper half sphere. [Hint: The Legendre polynomials $P_0(x)=1$, $P_1(x)=x$, and $P_2(x)=(3x^2-1)/2$.]



2. (7%, 7%, 6%)

(a) What are Green's first identity and Green's theorem?

(b) What are the requirements on the Green function for the Dirichlet and Neumann boundary conditions?

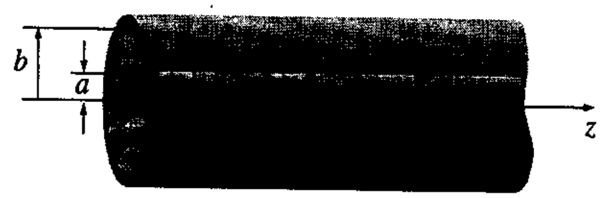
(c) For a point charge outside a grounded conducting sphere, find the Green function $G(\mathbf{x},\mathbf{x}')$ that satisfies Dirichlet boundary conditions. [Hint: the method of images.]

3. (10%, 10%) For an air-filled ($\varepsilon=\varepsilon_0$, $\mu=\mu_0$) coaxial waveguide with inner radius a and outer radius b as shown below, the electric and magnetic fields are:

$$\mathbf{E}(s,\phi,z,t)=\frac{E_0\cos(kz-\omega t)}{s}\hat{\mathbf{s}} \text{ and } \mathbf{B}(s,\phi,z,t)=\frac{E_0\cos(kz-\omega t)}{cs}\hat{\boldsymbol{\phi}}$$

(a) Find the time-averaged energy density u.

(b) Find the energy flow $\mathbf{S}$ (energy per unit area per unit time) along the line.



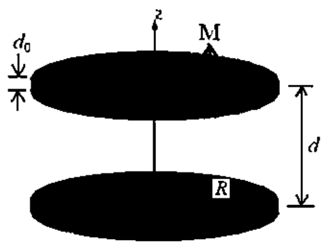
4. (10%, 10%) Consider two thin ferrite disks of radius R and thickness d_0, separated by a distance d ($>>d_0$). Assume the ferrite disks carry a uniform magnetization, $\mathbf{M} = M_r \hat{z}$.

(a) Find the bound surface currents $\mathbf{K}_b$ on the outer surfaces of the ferrite disk and the bound volume current density $\mathbf{J}_b$.

(b) Find the magnetic field at the midpoint of the central axis when $d=R$.

[Hint: The arrangement is similar to a Helmholtz coil.]

[Hint: $\mathbf{A}_M(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\nabla' \times \mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' + \frac{\mu_0}{4\pi}\oint_s \frac{\mathbf{M}(\mathbf{x}') \times \hat{\mathbf{n}}'}{|\mathbf{x}-\mathbf{x}'|} da'$].



5. (10%, 10%) The generalized dielectric constant.

Assume there are N molecules per unit volume and Z electrons per molecule. Each molecule contains f_j electrons with binding frequency ω_j and damping factor γ_j, where $Z=\Sigma f_j$. [Hint: Neglect the free electrons.]

(a) Find the polarization $\mathbf{P}$.

(b) Find the form of the complex dielectric constant.

Quantum Mechanics Qualification, Sep. 19, 2010.

You must provide the details or reasonings to justify your answers.

1. (9 +6%)[**Addition of angular momenta**]
   Work out the Clebsch-Gordan(CG) coefficients of
   (a) $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$
   (b) $1 \otimes 1 = 2 \oplus 1 \oplus 0$

2. (15%)[**time-dependant perturbation**]
   At $t = -\infty$ a particle is in its ground state in an 1D harmonic oscillator potential. At $t = 0$, a perturbation $V(x,t) = V_0 \hat{x} e^{-t/\tau}$ is turned on. Calculate to first order the probability that at $t = \infty$ the system will have made a transition to its first, second, and third excited states.

3. (6+8+6%)[**Identical Particles**]
   (a)What are Fermions, and what are Bosons? How do they differ from each other?
   (b) Write down the spatial and spinal wave function of the first excited state for two noninteracting electrons in an infinite potential well, $V(x) = 0$ for $0 \le x \le L$ and $V(x) = \infty$ elsewhere.
   (c) Similarly, write down the spatial wave function of the first excited state for two noninteracting spin-0 particles in the same potential well.

4. (10+10%)[**Quantum measurements**]
   (a) Measurement of an electron's spin along the z-axis(S_z) using a Stern-Gerlach apparatus gives the eigenvalue $-\hbar/2$. What is the probability that a subsequent measurement of the spin in the direction $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ yields $+\hbar/2$?
   (b) Measurement of an electron's spin along the axis $\hat{n}$ gives the eigenvalue $+\hbar/2$. What is the probability that a subsequent measurement of the spin along the x-axis yields $+\hbar/2$?

5. (5+10%)[**electron in a constant B-field**]
   An electron moves in a uniform magnetic field which is pointing to the z-direction ($\vec{B} = B\hat{z}$).
   (a) Evaluate
   $$\left[\hat{\Pi}_x, \hat{\Pi}_y\right],$$
   where $\hat{\Pi}_x = \hat{p}_x - e\hat{A}_x/c$, $\hat{\Pi}_y = \hat{p}_y - e\hat{A}_y/c$.
   (b) By comparing the Hamiltonian and the commutation relation obtained in (a) with those of the 1D simple harmonic oscillator problem, show that the energy eigenvalues are:
   $$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \left(\frac{|eB|\hbar}{mc}\right)\left(n + \frac{1}{2}\right)$$

where $\hbar k$ is the continuous eigenvalue of the $\hat{p}_z$ operator and n is a positive integer (including zero).

6. (10+5%) A particle whose wave function is given by

$$\psi(\vec{r}) = \left(\frac{1}{\sqrt{3}} Y_{11}(\theta,\phi) - \frac{\sqrt{7}}{\sqrt{6}} Y_{1,-1}(\theta,\phi) + \frac{1}{\sqrt{2}} Y_{10}(\theta,\phi) \right) f(r)$$

where $f(r)$ is a normalized radial function, $\int_0^\infty r^2 f^2(r) dr = 1$.
(a) What are the expectation values of $\hat{L}^2$, $\hat{L}_z$, and $\hat{L}_x^2$?
(b) What is the expectation value of an operator $\hat{V}(\theta) = 2\cos^2\theta$ in this case?

You may find the following information useful:

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta \,,\; Y_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$