Classical Electrodynamics (2020 March)

Please use SI unit system. If not, please indicate the system you use.

1. A dielectric sphere with dielectric constant $\varepsilon > 1$ is placed in a uniform electric field. Show the change in the uniform electric field by drawing the field lines around and inside the dielectric sphere. (10%)

2. A point charge $q$ is placed at distance $d$ from an infinite plane conductor held at zero potential. What is the surface charge density induced on the plane. (10%)

3. What is the equation of energy conservation (Poynting's Theorem) for a system of moving charges in a vacuum. (15%)

4. Derive the wave equation for electromagnetic fields from the Maxwell equations. (10%)

5. What is the Maxwell stress tensor? Express the conservation of momentum with the Maxwell stress tensor. (10%)

6. What are the electric and magnetic fields generated by a point charge moving with a constant velocity $v$ (Note that $|\mathbf{v}|$ can be close to the speed of light). (10%)

7. If $\mathbf{E}$ and $\mathbf{B}$ are perpendicular in the laboratory and $E = 2B$, can you find a reference frame such that the field is pure electric or magnetic? If yes, what is the velocity of the reference frame relative to the laboratory? (10%)

8. Explain the following terms.

(a) Kramer-Kroning Relation (4%)

(b) Skin depth (4%)

(c) Plasma frequency (4%)

(d) Stokes Parameters (5%)

(e) Group velocity (4%)

(f) Snell's law (refraction) (4%)

Qualifying Examination – Statistical Mechanics

Spring, 2020

1. (Fermi gas, 30 points) For $N$ ideal electrons of individual mass $m$ in volume $V$,
   (a) Derive the Fermi energy $E_F$.
   (b) Derive the degenerate pressure at zero temperature. How will your result change if $N/V$ is high enough, as in dwarf stars, to become relativistic?
   (c) Show that $-\frac{dn_{FD}(\varepsilon)}{d\varepsilon}$, where $n_{FD}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$ denotes the Fermi-Dirac distribution and $\beta \equiv \frac{1}{k_B T}$, can be treated as a probability distribution $P(\varepsilon)$ that obeys: (i) $\int_0^\infty P\, d\varepsilon = 1$ , (ii) $\int_0^\infty (\varepsilon - \mu) P\, d\varepsilon = 0$ , (iii) $\int_0^\infty (\varepsilon -$ $\mu 2Pd\varepsilon{\approx}kBT2$ at temperature $T{\ll}EF$.
   (d) Derive the expression for chemical potential $\mu$ at temperature $T \ll E_F$.
   (e) Derive the expression for specific heat $C_V$ at temperature $T \ll E_F$.

2. (Ideal and non-ideal classical gas, 25 points)
   (a) Find the Helmholtz free energy $F$ for $N$ ideal classical particles of individual mass $m$ in volume $V$ and at temperature $T$. (Note: Remember to include $\frac{1}{N!}$ to prevent the Gibbs paradox. Given the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ )
   (b) Derive the state equation $PV=Nk_BT$ by $P = -\left.\frac{\partial F}{\partial V}\right|_{T,N}$.
   (c) Consider the fact that real particles (i) repel each other at close range with a hard volume $v$ and (ii) attract each other at long range by adding a potential energy term[1] $-\alpha \left(\frac{N}{V}\right)^2 V$ to the free energy. Show that these modifications lead us to the Van der Waals equation: $\left[P + \alpha \left(\frac{N}{V}\right)^2\right](V - Nv) = Nk_B T$.
   (d) Take two containers of different gas and open the partition to allow them to mix. What is the entropy change after the mixing? But, had the two gases been the same, we do not expect any entropy change! What happened?

3. (Ising model, 15 points) The Ising Hamiltonian looks like $H = -J \sum_{\langle i,j \rangle} S_i S_j$ where $\langle i,j \rangle$ denotes sites $i$ and $j$ are nearest neighbors and $S = \pm 1$. Use $q$ to denote the number of nearest neighbors for each spin.
   (a) Derive the expression for Curie temperature $T_C$ in the mean field approximation

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[1] The parameter, $\alpha$, describes the strength of pairwise attractive force between particles, while $\left(\frac{N}{V}\right)^2$ comes from the joint probability. Finally, the $V$ factor comes from integrating the energy density $-\alpha \left(\frac{N}{V}\right)^2$ over the whole volume to obtain the potential energy.

(Hint: $T_C$ refers to the temperature below which spins become parallel aligned.)

(b) Solve the Ising model in one dimension exactly and show that $T_C = 0$, i.e., no phase transition to a magnetic state at any temperature for an Ising chain.

4. (Free bosons, 15 points)

Consider now a number conserved Bose gas with energy dispersion $\varepsilon_{\mathrm{p}} = C|\boldsymbol{p}|^a$:

(a) Show that there will be a Bose-Einstein condensation if d>a

(b) Show that the critical temperature scales with the total number of particles, $N$, as $T_c \propto N^{a/d}$. (Hint: you do not need to do the full calculation, and may use dimensional argument for the density of states first).

5. (Random walk, 15 points)

(a) A drunkard is kicked out of the pub. Being fully intoxicated, he walks forward and backward with equal probability. According to the senior high-school math, the probability of finding him at position $n$ after $N$ steps is $P(n, N) =$

$C_{N_\rightarrow}^{N} \left(\frac{1}{2}\right)^{N_\rightarrow} \left(\frac{1}{2}\right)^{N_\leftarrow}$ where $N_\rightarrow$ and $N_\leftarrow$ represent the number of steps forward and backward, respectively. Naturally,

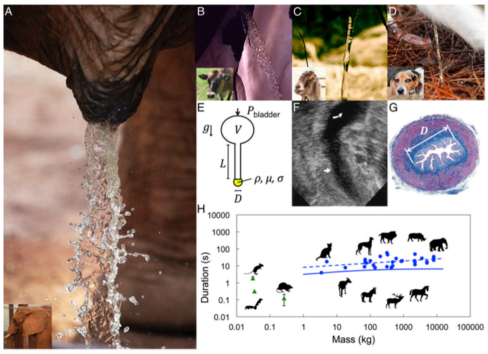
$$N = N_\rightarrow + N_\rightarrow \text{ and } n = N_\rightarrow - N_\rightarrow \qquad (1)$$

Assume $N \gg n \gg 1$ so that the Stirling formula can be used to approximate all large factorials: $lim_{N\gg1} N! \approx NlnN - N$. Show that $P(n, N)$ can be reduced to the Gaussian distribution: $P(n, N) \sim \frac{1}{\sqrt{N}} \exp\left(-\frac{n^2}{4N}\right)$.

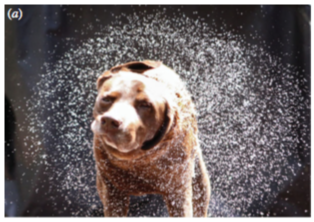
Qualification examination (February 2020)

- Classical Dynamics (20 points each) -

1. (Application) Be creative at using knowledge of classical dynamics to explain

   (a) A study[1] showing that nearly all mammals with weight over 3kg take about the same amount of time, 21 seconds, to urinate was awarded the 2015 Ig Nobel prize（搞笑諾貝爾獎）at Harvard.

   (b) In cold wet weather, mammals face hypothermia（失溫）if they cannot dry themselves rapidly. Scientists at GIT found[2] that the frequency $f$ animals of mass $M$ use to oscillate and shake the water off their bodies scales as $f \sim M^{-0.22}$.

2. (Over-damped motion) A pendulum of length $\ell$ with a bob (擺錘) of mass $m$ at its end moves through oil that retards its motion with a resistive force, $\alpha(\ell\dot{\theta})$. Denote the initial displacement and angular velocity by $\theta_0 > 0$ and $\omega_0$. Please find $\theta(t)$? Can $\theta$ ever become negative? If yes, what is the condition for this to happen and what is the maximum number of times $\theta$ can change sign?

3. (Gravitation) Derive Kepler's third law: $\tau^2 \propto a^3$ where $\tau$ is the period and $a$ the semi-major axis (長軸半長). *Hint*: (a) draw an elliptic orbit with semi-major and minor axes $a, b$ and put the sun on one of its focus (焦點), (b) denote the speed of planet at perihelion (近日點) and aphelion (遠日點) by $v_p, v_a$, (c) write down the conservations of angular momentum $\ell$ and mechanical energy at perihelion and aphelion, (d) given that the area of ellipse equals $ab\pi$, relate $\tau$ to $\ell$.

4. (Calculus of variations) Find the curve $y(x)$ of length $\ell$ bounded by the $x$-axis on the bottom that passes through the points $(\pm a, 0)$ and encloses the largest area. What is the value of $a$ determined by the problem?

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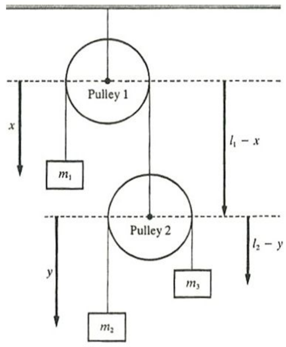
[1] P. J. Yang, J. Pham, J. Choo, and D. L. Hu, "Duration of urination does not change with body size", PNAS 111, 11932 (2014). https://www.bbc.com/news/science-environment-34278595

[2] A. K. Dickerson, A. G. Mills, and D. L. Hu, "Wet mammals shake at tuned frequencies to dry", Interface **9**, 3208 (2012).

5. Determine the equations of motion of

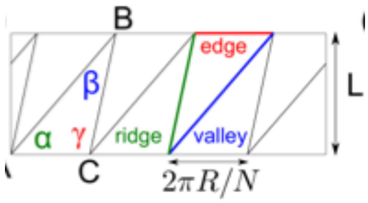
(a) A disk of mass $m$ and radius $r$ rolls down without slipping on an inclined plane of angle $\theta$ and mass $M$ that can move frictionless on a horizontal plane.

(b) The double pulley system shown on the right.




6. (Bonus 10 points) When you twist a paper cylinder with length $L$ shorter than its diameter $2R$, $N$-pairs of regular valleys（凹摺）and ridges（凸摺）appear automatically on its surface.


(a) Given $R, L,$ and $N \gg 1$, determine angles $\alpha, \beta$ from geometry, such as the sum of interior angles in a triangle equals $\pi$, law of sines (正弦定理), and relate each of the interior angle of the regular $N$ polygon (formed on both edges of the cylinder) is constructed by $\alpha, \beta, \gamma$.




(b) Assuming $N \gg 1$, check your derivations in (a) and see if $L > 2R$ will violate the fact that $\sin\theta$ should never exceed unity? In other words, the pattern of creases on a long cylinder can never be regular.

Qualifying Examination - Quantum Mechanics

Spring, 2020

1. (Terminology explanation, 30 points)

   Please explain the following terminologies in QM (0,3-0.5 page each)

   (a) Addition of Angular Momentum

   (b) Clebsch-Gordan coefficients

   (c) Hyperfine interaction

   (d) Path Integral Method

   (e) Dirac Equation

   (f) Variational Method

2. (Infinite square well potential in 3D, 5+5+10 points)

   Consider a particle of mass m, confined in a 3D infinite equation well potential, $V(r)=0$ as $r=|\mathbf{r}|<a$, and infinite for $r>a$.

   (a) Write down the 3D static Schrodinger equation in spherical coordinate with proper boundary condition for the wavefunction $\psi(r,\theta,\phi)$

   (b) Using the separation of variables, $\psi(r,\theta,\phi)=R_{nl}(r)Y_{lm}(\theta,\phi)$, to derive the associated equations.

   (c) Using $\chi_n(r)\equiv rR_{n0}(r)$ to derive the equation for $l$=0 state, and solve the eigen-energies by the boundary conditions.

3. (Spin, 10 points)

   For 87 Rb atom, the nuclear spin is I=3/2 and has one electron in the s-state with spin S=1/2. The Hyperfine coupling is like $\boldsymbol{H}=\boldsymbol{A}\boldsymbol{I}\cdot\boldsymbol{S}$, where I and S are nuclear spin and electron spin operators respectively.

   (a) Calculate the hyperfine splitting energy.

   (b) What is the quantum number for the total angular momentum? Express these angular momentum eigenstates in terms of original spin and nuclear spin states.

4. (Time-independent perturbation Theory, 20 points)

   Following the question above with an additional magnetic field is applied, the additional term of the Hamiltonian is like $H_1=-\mu_S\mathbf{S}\cdot\mathbf{B}-\mu_I\mathbf{I}\cdot\mathbf{B}$.

(a) Calculate the full Hamiltonian in the total spin basis.

(b) Using perturbation theory, calculate the leading order correction of eigenstate energies.

5. (Time-dependent perturbation theory, 20 points)
At $t = -\infty$ a particle is in its ground state in an 1D harmonic oscillator potential. At $t = 0$, a perturbation $V(x,t) = V_0 \hat{x} e^{-t/\tau}$ is turned on. Calculate to first order the probability that at $t = \infty$ the system will have made a transition to its first, second, and third excited states.