Qualification Exam for PhD Candidates (Classical Mechanics, Sep. 2016)

1. (10%) What is the total cross section for the elastic scattering of a beam of particles of radius r from a fixed solid sphere with radius R?

2. (15%) A proton with relativistic energy E hits a proton at rest to produce a pair of proton-antiproton

$$p + p \rightarrow p + p + (p + \bar{p}).$$

Use special relativity to calculate the minimum energy of the incident proton. The mass of a proton is m.

3. (15%) Write down (without proof) the most general solution of the wave equation

$$\frac{\partial^2 F(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F(x,t)}{\partial t^2} = 0$$

and explain the physical meaning of the solution.

4. (15%) Two particles with the same mass m move in one dimension at the junctions of three springs, as shown in the figure 1. The springs all have the same unstretched length. The spring constant in the middle spring is $4k$. Two other springs have the same spring constant k. Find the eigenfrequencies and normal modes of the system.

5. (15%) Consider the motion of a particle with mass m in a central force field and use the spherical polar coordinates (r, θ, ϕ). The potential energy is $V(r)$. (a) Write down the Lagrangian of the system. (b) Derive the Hamiltonian from the Lagrangian. (c) Write down the Hamilton's equations. (Do not solve the equations).

6. (15%) A rocket is fired at 60^0 to the local vertical line with an initial speed $v_0 = \sqrt{GM/R}$ where M is the mass of earth and R is its radius (as shown in figure 2). Find the maximum distance from the center of the earth. The effect of rotation of earth can be neglected.

7. (15%) A particle of mass m moves in one dimension with potential energy $A|x|$ where A is a positive constant. By the method of action-angle variables to find the expression for the period of motion as a function of the particle's energy E.

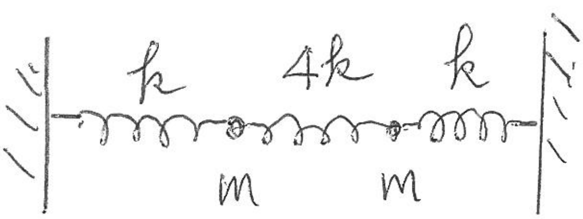

Fig. 1

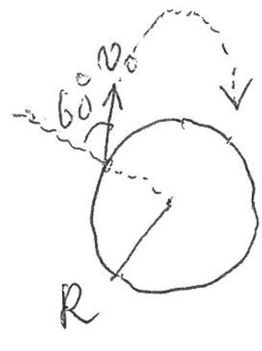

Fig. 2

Qualifying Exam: **Classical Electrodynamics** Sep. 2016

1. (30%) Explain the following terms qualitatively and quantitatively.

(a) Rayleigh scattering (5%)

(b) Perfect conductor and super conductor (5%)

(c) Plasma frequency (5%)

(d) Lorentz and Coulomb gauges (5%)

(e) Liénard-Wiechert potentials (5%)

(f) Explain why the sky is blue at noon and red at sunset. (5%)

2. (10%, 10%) Green function

(a) What are Green's first identity and Green's theorem?

(b) For a point charge q **outside** a grounded conducting spherical shell of radius a, find the Green function $G(\mathbf{x},\mathbf{x}')$ that satisfies Dirichlet boundary condition. [Hint: the method of images.]

3. (10%, 10%) Two concentric shells of inner and outer radii a and b ($a<b$), respectively. The outer shell is grounded $V(b,\theta)=0$.

(a) If $V(a,\theta)=V_0\cos\theta$, find the potential at $r<a$, $a<r<b$, and $r>b$.

(b) If $V(a,\theta)=V_0\cos^2\theta$, find the potential everywhere between the shells ($a<r<b$). [Hint: use Legendre polynomials $P_0(x)=1,\ P_1(x)=x,$ and $P_2(x)=(3x^2-1)/2$].

4. (10%, 10%) A particle with mass m and charge $+q$ starts from rest at the origin under a uniform electric field $\mathbf{E}=E\ \hat{\mathrm{y}}$ and a uniform magnetic field $\mathbf{B}=B\ \hat{\mathrm{z}}$. Neglecting earth's gravity, the particle is accelerated to the +y-direction by the E field, but the magnetic force bends it to the right resulting an interesting phenomenon called $\mathbf{E}\times\mathbf{B}$ drift. At point P, the particle reaches its maximum height called $y_{\max}$. Determine the curve of the trajectory and express $y_{\max}$ in terms of m, q, E, B.

5. (10%) The transformations between two inertial systems S and S' are $x'=\gamma_v(x-vt)$ and $t'=\gamma_v(t-vx/c^2)$. Show that when $\Delta t=0$, $\Delta x=\Delta x'/\gamma_v$; but when $\Delta t'=0$, $\Delta x'=\Delta x/\gamma_v$. Explain why the length relations depend on simultaneity.

Quantum Mechanics Qualification
Fall, 2016

Problem 1 Answer the following questions *briefly*

(a) 16% Explain the following terms briefly: (i) spontaneous emission (ii) Aharonov-Bohm effect (iii) selection rules (iv) Kramer degeneracy

(b) 4% What are generators of spatial translations ?

(c) 5% Let the minimum uncertainty in the position measurement of a single electron without ambiguity be δx_0. What is the ratio (magnitude) of δx_0 to the Bohr radius?

(d) 5% Estimate the ground state energy (in units of eV) of a two-electron atom with the nuclear charge Z by using the uncertainty relation.

Problem 2 Consider a perturbation $H' = \alpha x^4$ to the motion of the harmonic oscillator. Here α is a positive number. Let m and ω be the mass and the natural frequence of the oscillator. Answer the following questions:

(a) 10% Show that the first-order correction to the nth unperturbed eigen-energies are

$$E_n^1 = (3\hbar^2\alpha)/(4m\omega^2)[1 + 2n + 2n^2],]. \tag{1}$$

No matter how small α is, the perturbation expansion will break down for some large enough n. Why?

(b) 10% If we make the perturbation time-dependent, $H' = \alpha x^4 e^{-t^2/\tau^2}$, between $t = -\infty$ and $t = +\infty$. What is the probability that the oscillator originally in the ground state ends up in the state $| - n\rangle$ at $t = +\infty$? You might find the following identity useful. $\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \sqrt{\pi/a} e^{b^2/4a}$.

Problem 3

(a) 5% Find $\mathbf{L}\cdot\mathbf{S}\, Y_1^{-1}(\theta,\phi)$ in terms of $Y_l^m(\theta,\phi)$, $|+\rangle$, and $|-\rangle$, where $\mathbf{L}$ are the orbital angular momentum operator and $\mathbf{S}$ are the spin operators.

(b) 8% Suppose that a particle has a magnetic dipole moment $\mu = g\mu_b \mathbf{J}$, where g is the g-factor, $\mu_b = e\hbar/2m$ is the Bohr magneton, and $\mathbf{J}$ is the total angular momentum.

(i) If the particle is placed in a uniform magnetic field $\mathbf{B}$ with the Hamiltonian being given by $H = -\mu \cdot \mathbf{B}$, find the equation of motion for the average total angular momentum $< \mathbf{J} >$.

(ii) Now consider a special case when $\mathbf{J} = \mathbf{S}$ with $s = 1/2$. Suppose that the magnetic field is $\mathbf{B} = B_0\hat{z}$ with B_0 being a constant. At $t = 0$, the spin of the particle is measured to be pointing along the positive y-axis. Find $\langle S_x \rangle$ and $\langle S_z \rangle$ at $t > 0$.

(c) 7% Consider a system of two particles with spins $s_1 = 1$ and $s_2 = 1/2$. By considering the addition of two spins, find all possible eigenvalues to square of the difference of spin operators $(\mathbf{S}_1 - \mathbf{S}_2)^2$.

Problem 4

(a) 7% Write down the spatial and spinal wave function of the first excited state for two noninteracting electrons in an infinite potential well, $V(x) = 0$ for $-a < x < a$ and $V(x) = \infty$ elsewhere.

(b) 7% Similarly, write down the spatial wave function of the first excited state for two noninteracting spin-0 particles in the same potential well.

Problem 5

Consider a particle of mass m in a central scattering potential $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ elsewhere. Here V_0 is a positive constant.

(a) 6% Find the minimum V_0 that there is at least one bound state for the particle when the angular moment vanishes, i.e., $l = 0$.

(b) 10% Suppose that the particle is incident with the momentum $\hbar k$. Using the Born approximationm, find the scattering amplitude for the particle and evaluate the total scattering cross section in the limit of very low incident energy ($ka \ll 1$).

Qualifying Exam for Statistical Mechanics

(Fall, 2016)

1.(20%) Fundamental concepts. Please EXPLAIN the following basic concepts in statistical mechanics (not just write down equations/formula without elaborated details):

(a) Gibbs paradox (and its relationship with quantum statistics)

(b) Ergodic hypothesis (and its relation with thermal equilibrium)

2.(20%) Harmonic Oscillator. A one-dimensional quantum harmonic oscillator (whose ground state energy is $\varepsilon_0 = \hbar\omega/2$) is in thermal contact with a heat reservoir at temperature T. Determine the average energy $u(T) = \langle\varepsilon\rangle$ of the harmonic oscillator in thermal equilibrium.

3.(20%) Fermi pressure. For free Fermi gas, the pressure and the internal energy are related $P = 2U/(3V)$, where U is the internal energy and V is the volume of the gas. **(a)** Explain *Fermi pressure* in details. **(b)** Obviously, the Fermi pressure violates the ideal gas law $PV = NkT$. Suppose the density is n and the particle mass is m. Write down the criterion for the idea gas law to be approximately valid in the Fermi gas.

4.(20%) Black hole entropy. According to Bekenstein and Hawking, the entropy of a black hole is proportional to its surface area A,

$$S = \frac{k_B c^3}{4G\hbar} A,$$

where G is the Newton constant and c is the speed of light. **(a)** Calculate the escape velocity at a radius R from a mass M using classical mechanics. Find the relationship between the radius and mass of a black hole by setting the escape velocity to the speed of light c. **(b)** The internal energy of the black hole is given by $E = Mc^2$. Find the temperature of the black hole in terms of its mass.

5.(20%) Gas adsorption. A surface with N_0 adsorbing centers has average $\langle N\rangle < N_0$ gas molecules adsorbed on it. The partition function of a single molecule is $a(T)$ and interactions between the adsorbed molecules can be neglected. Find the chemical potential μ of the system in thermal equilibrium by constructing the grand partition function.