Qualifying Exam: **Classical Electrodynamics** Feb. 21, 2016

1. (30%) Explain the following terms qualitatively and quantitatively.

(a) Dirichlet and Neumann boundary conditions (5%)

(b) Laplace and Poisson equations (5%)

(c) Maxwell stress tensor (5%)

(d) Lorentz and Coulomb gauges (5%)

(e) Kramers-Kronig relations (5%)

(f) Synchrotron radiation (5%)

2. (10%, 10%) Two concentric conducting shells of inner and outer radii a and b ($a<b$), respectively. The outer shell is connected to a given potential V, while the inner shell is grounded $V(a,\theta)=0$.

(a) If $V(b,\theta)=V_0$ (constant), find the potential at $r<a$, $a<r<b$, and $r>b$.

(b) If $V(b,\theta)=V_0\cos^2\theta$, find the potential everywhere between the shells ($a<r<b$). [Hint: use Legendre polynomials $P_0(x)=1,\ P_1(x)=x,$ and $P_2(x)=(3x^2-1)/2$].

3. (10%, 10%) Green function

(a) What are Green's first identity and Green's theorem?

(b) For a point charge q **inside** a grounded conducting spherical shell of radius a, find the Green function $G(\mathbf{x},\mathbf{x}')$ that satisfies Dirichlet boundary condition. [Hint: the method of images.]

4. (10%, 10%) A linearly polarized plane wave $\mathbf{E}(\mathbf{x},t)=\mathrm{Re}[E_0\boldsymbol{\varepsilon}_{\parallel}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}]$ is incident on a plane surface between two media of different dielectric properties, where $\boldsymbol{\varepsilon}_{\parallel}$ is the electric field parallel to the plane of incidence. The media below and above the plane $z=0$ have the permeabilities and permittivities $\mu_0,\ \varepsilon_0$ and $\mu_0,\ \varepsilon_r\varepsilon_0$. The corresponding index of reflections are 1 and $n\equiv\sqrt{\varepsilon_r}$, respectively.

(a) When the angle of incidence is equal to Brewster's angle, $i_B=\tan^{-1}(n)$, find the reflected and refracted (transmitted) electric fields.

(b) Show that the incident intensity is equivalent to the transmitted intensity. [Hint: Use the (Complex) Poynting vector.]

5. (10%) Show that $|\mathbf{E}|^2-|\mathbf{B}|^2$ is a Lorentz scalar.

Qualifying Exam for Statistical Mechanics

(Spring, 2016)

1.(20%) Fundamental concepts. Please EXPLAIN the following basic concepts in statistical mechanics (not just write down equations/formula without elaborated details):

(a) Fermi statistics and Fermi pressure

(b) Gibbs paradox and its relationship with quantum statistics

2.(20%) Density matrix. Consider a free particle of mass m described by the Hamiltonian $H = (p_x^2 + p_y^2 + p_z^2)/2m$. In thermal equilibrium, the density matrix operator for the particle is $\rho(T)$ at temperature T. Calculate the density matrix operator in coordinate representation $\langle \boldsymbol{r}|\rho(T)|\boldsymbol{r}'\rangle$.

3.(20%) Free Fermions and Bosons. **(a)** In the grand canonical ensemble, compute the average particle number $\langle n_B(\epsilon)\rangle$ for free bosons at the energy level $E = \epsilon$. Compute the same quantity $\langle n_F(\epsilon)\rangle$ for free fermions. **(b)** Compute the entropy S for both free bosons and fermions, expressing it in terms of $\langle n_B(\epsilon)\rangle$ and $\langle n_F(\epsilon)\rangle$ respectively.

4.(20%) Equipartition theorem. Consider a system with the Hamiltonian $H(q_i, p_i)$, where $i = 1, 2, ..., f$. In the high-temperature limit (classical limit), the summation over all quantum states can be approximated as integration over all degrees of freedom q_i, p_i. **(a)** Compute the following average quantity $\langle q_i\,(\partial H/\partial q_j)\rangle$ in the canonical ensemble. (Hint: you may start with computing the thermal average $\langle q_i\rangle$ in the classical limit.) **(b)** Suppose the Hamiltonian of the statistical system is quadratic in both p_i and q_i,

$$H = \sum_{i=1}^{f} A_i p_i^2 + \sum_{i=1}^{f} B_i q_i^2,$$

where A_i and B_i are constants. Compute the average energy of the statistical system $U(T) \equiv \langle H\rangle$ at temperature T.

5.(20%) Bose-Einstein condensation. Consider a non-interacting Bose gas of N particles with energy dispersion $E(\boldsymbol{p}) = C|\boldsymbol{p}|^z$. **(a)** Show that the existence of Bose-Einstein condensation requires $d > z$. **(b)** Show that the critical temperature scales with the total number of particles, $T_c \propto N^{z/d}$. (Hint: you do not need to do the full calculation, and may use dimensional argument for the density of states first).

Qualification Exam for PhD Candidates (Classical Mechanics, Feb. 2016)

1. (10%) What is the total cross section for the elastic scattering of a beam of particles of radius r from a fixed solid sphere with radius R?

2. (15%) A photon hits an electron at rest, producing an electron-positron pair:
$$\gamma + e^- \to e^- + e^+ + e^-.$$
Use special relativity to calculate the minimum energy of the incident photon.

3. (15%) Write down (without proof) the most general solution of the wave equation
$$\frac{\partial^2 F(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F(x,t)}{\partial t^2} = 0$$
and explain the physical meaning of the solution.

4. (15%) Consider the elastic scattering of two particles with masses M and m where $M > m$. The speed of M is V and the particle m is at rest. Find the maximum scattering angle θ of M after scattering.

5. (15%) Consider two coupled oscillators with the same mass m and spring constant k. The potential energy is
$$V = k(x_1^2 + x_2^2)/2 + \epsilon x_1 x_2$$
where ϵ is the coupling constant and x_i are the coordinates of the oscillators. Find the angular frequencies of vibration.

6. (15%) Consider the motion of a particle with mass m in a central force field and use the spherical polar coordinates (r, θ, ϕ). The potential energy is $V(r)$. (a) Write down the Lagrangian of the system. (b) Derive the Hamiltonian from the Lagrangian. (c) Write down the Hamilton's equations. (Do not solve the equations).

7. (15%) A rocket is fired at 60^0 to the local vertical line with an initial speed $v_0 = \sqrt{GM/R}$ where M is the mass of earth and R is its radius. Find the maximum distance from the center of the earth. The effect of rotation of earth can be neglected.

Quantum Mechanics Qualification
Spring, 2016

Problem 1 Answer the following questions *briefly*

(a) 16% Explain the following terms briefly: (i) collapse of state (ii) Aharonov-Bohm effect (iii) Clebsch-Gordan coefficients (iv) scattering length

(b) 4% What is the generator for the translation in time?

(c) 5% Find Hermitian conjugates of the operators: $2x\frac{d}{dx}$ (in terms of x and $\frac{d}{dx}$), and $2^{-i|1\rangle\langle 2|}$ (Here $|1\rangle$ and $|2\rangle$ are two states in the Hilbert space).

(d) 5% Let $\hat{\mathrm{P}}$ and $\hat{\mathrm{L}}$ be the three dimensional momentum and orbital angular momentum operators. Consider the operator

$$\exp\left(\frac{i\hat{\mathrm{J}}_z\phi}{\hbar}\right)\hat{\mathrm{P}}_y\hat{\mathrm{L}}_x\exp\left(-\frac{i\hat{\mathrm{J}}_z\phi}{\hbar}\right).$$

Here $\hat{\mathrm{J}}$ is the total angular momentum operator. Express the above operators in terms of $\hat{\mathrm{P}}_x$, $\hat{\mathrm{P}}_y$, $\hat{\mathrm{L}}_x$, $\hat{\mathrm{L}}_y$, and ϕ.

Problem 2 Consider two particles of the same mass m in one dimension with coordinates being denoted by x and they are connected by a spring with spring constant k. Suppose that the total momentum of the system is p, answer the following questions:

(a) 10% find all possible total energies for the following cases: (i) two particles are different (ii) two particles are identical fermions (iii) two particles are identical bosons.

(b) Suppose that these two particles carry charges of q and $-q$ and are placed in a constant electric field $E = -\frac{d\phi(x)}{dx}$. Let $\phi(0) = 0$.

(i) **5%** Find all energy eigenvalues.

(ii) **7%** If initially, the system of these two particles is in the ground state of the Hamiltonian (with $p = 0$) without being perturbed by the electric field (i.e., $E = 0$), find the probability of finding the system in the ground state in the presence of the electric field.

(iii) **8%** Following (ii), suppose that the electric field E is switched on from the time $t = -\infty$ in the following way

$$E = \frac{E_0}{1 + (t/\tau)^2},$$

where τ is a positive constant and E_0 is a constant as well. If the system of these two particles is in the ground state at $t = -\infty$, find the probability that the system is in one of the excited states at $t = \infty$ to order of E_0^2.

Problem 3

(a) 5% Find $\mathbf{L} \cdot \mathbf{S}\,(Y_1^1(\theta,\phi) + 3Y_1^0(\theta,\phi))\,|+\rangle$ in terms of $Y_l^m(\theta,\phi)$, $|+\rangle$, and $|-\rangle$, where $\mathbf{L}$ are the orbital angular momentum operator and $\mathbf{S}$ are the spin operators.

(b) 5% A beam of unploarized spin-1/2 particles, moving along the y-axis, is incident on two collinear Stern-Gerlach apparatuses, the first with $\mathbf{B}$ along the z axis and the second with $\mathbf{B}$ along the z' axis, which lies in the x-z plane at an angle $\pi/3$ relative to the x axis. Both apparatuses transmit only the uppermost beams, what fraction of particles leaving the first will leave the last?

(c) 10% Consider a system of two particles with spins $s_1 = 1$ and $s_2 = 1/2$. By considering the addition of two spins, find all possible eigenvalues to square of the difference of spin operators $(\mathbf{S}_1 - \mathbf{S}_2)^2$.

Problem 4 20% Consider the elastic scattering between two neutrons. Suppose that the interaction potential can be approximated by

$$V(r) = V_0 \vec{S}_1 \cdot \vec{S}_2 \frac{e^{-r/a}}{r},$$

where r is the distance between two neutrons, $V_0 > 0$, $a > 0$, and $\vec{S}_i$ $(i = 1,2)$ are the spin vector operators of the two neutrons. In the lab frame, the scattering is set up in the way that one neutron is initially at rest, while the other one is incident with a momentum $\hbar\mathbf{k}$. Both neutrons are not polarized. Let the mass of each neutron be m. Use the Born approximation to calculate the differential cross section $\sigma(\theta,\phi)$ in the center of mass (CM) frame. What is the differential cross section in the lab frame?