Qualifying Examination – Statistical Mechanics

Fall, 2015

1. (Fundamental concepts in statistical mechanics, 30 points)
Please EXPLAIN (not just describe or write equations/formula) the following terminologies in Statistical Mechanics.
(a) Canonical Ensemble and Grand Canonical Ensemble
(b) Fermi statistics and Fermi pressure
(c) Gibbs paradox and its relationship with quantum statistics

2. (Entropy, 20 points)
Calculate the entropy of the following three-dimensional systems consisting of N atoms with atomic weight m and at temperature T?
(a) (5 pts) A crystal. At very low temperature, the interaction between these atoms can be neglected. Assume the ground state of each atom is doubly degenerate.
(b) (5 pts) An ideal classical gas (free moving in space) with spin 2.
(c) (10 pts) An ideal Fermi gas (s=1/2) with T much less than the Fermi temperature.

3. (Two-level system, 20 points)
Consider a system of N distinguishable particles, which have two energy levels, $E_0=-\mu B$ and $E_1=\mu B$, for each particles. Here μ is magnetic moment and B is magnetic field. The particles populate the energy levels according to the classical distribution law.
(a) (10 pts) Calculate the average energy of such system at temperature T, and
(b) (5 pts) the specific heat of the system.
(c) (5 pts) Calculate the magnetic susceptibility.

4. (Ideal Fermi gas in 3-D, 30 points)
(a) Considering an ideal Fermi gas of spin S=5/2 at zero temperature, calculate the total internal energy by summing the electron energy all the way up to the Fermi energy.

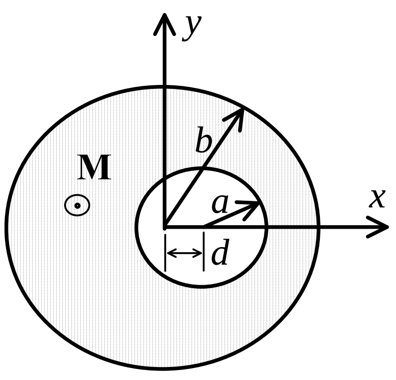
(b) Calculate the Fermi Pressure, P.

(c) Derive the Pauli magnetic susceptibility in the limit of zero magnetic field.

1. (20%) Explain the following terms qualitatively and quantitatively.
   (a) Plasma frequency (5%)
   (b) Lorentz gauge and Coulomb gauge (5%)
   (c) Retarded Green function (5%)
   (d) Synchrotron radiation (5%)

2. (10%, 10%) Green theorem and application
   (a) Use the Green's 1st identity to prove the uniqueness theorem for the Poisson's equation with Dirichlet boundary condition.
   (b) For a point charge q **outside** a grounded conducting spherical shell of radius a, find the Green function $G(\mathbf{x},\mathbf{x}')$ that satisfies Dirichlet boundary condition. [Hint: the method of images.]

3. (10%, 10%) A long cylindrical bar magnet of radius b has a hole of radius a bored parallel to, and centered a distance d from the cylinder axis. The magnetization $\mathbf{M}$ is uniform throughout the remaining magnet and is parallel to the axis.
   (a) Find the volume current density $\mathbf{J}_M$ inside the magnet and surface current density $\mathbf{K}_M$ in the hole.
   (b) Find the magnitude and the direction of the magnetic-flux density in the hole.
   [Hint: Use Ampere's law and the principle of superposition].



4. (10%, 10%) An ideal circular parallel plate capacitor of radius a and plate separation $d \ll a$ is connected to a current source by axial leads. The current in the wire is $I(t) = I_0 \cos \omega t$. [Hint: Calculate to the zeroth order in powers of the frequency and $\omega a/c \ll 1$]

(a) Calculate the electric and magnetic fields between the plate and neglecting the effects of fringing fields.

(b) Show that the equivalent series circuit has $C \simeq \pi\varepsilon_0 a^2/d$ and $L \simeq \mu_0 d/8\pi$.

5. (10%, 10%) A circularly polarized plane wave $\mathbf{E}(\mathbf{x},t) = E_0(\boldsymbol{\varepsilon}_{||} + i\boldsymbol{\varepsilon}_{\perp})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$ is incident on a plane surface between two media of different dielectric properties, where $\boldsymbol{\varepsilon}_{||}$ and $\boldsymbol{\varepsilon}_{\perp}$ are the electric field parallel and perpendicular to the plane of incident, respectively. The media below and above the plane $z = 0$ have the permeabilities and permittivities μ_0, ε and μ_0, ε', respectively.

(a) When the angle of incident is equal to Brewster's angle, $i_B = \tan^{-1}(n'/n)$, does the reflected radiation polarize linearly, elliptically, or circularly? Explain in details.

(b) When the wave is incident normally $i = 0$, find the refracted (transmitted) wave $\mathbf{E}'(\mathbf{x},t)$.

Qualification Exam for PhD Candidates (Classical Mechanics, September 2015)

1. (15%) Two particles move about each other in circular orbits under the influence of gravitational forces, with a period T_0. Their motion is suddenly stopped and then allowed to fall into each other. They collide after a time T_1. Calculate the ratio T_1/T_0.

2. (10%) What is the total cross section for the elastic scattering of a beam of particles of radius r from a fixed solid sphere with radius R?

3. (15%) Calculate the minimum total energy of a proton which hits another proton at rest to produce an anti-proton in the process

$$p + p \to p + p + p + \bar{p}.$$

   The rest energies of proton and anti-proton are 938 MeV, but you may use 1 GeV in your calculation.

4. (15%) The Hamiltonian of a harmonic oscillator is

$$H = \frac{p^2 + m^2\omega^2 q^2}{2m} = E.$$

   (a) Write down the Hamilton-Jacobi equation (do not solve it). (b) Use the method of action-angle variables to calculate the frequency of oscillation.

5. (15%) Consider a damped oscillator. An external driving force is applied to the oscillator The equation of motion is

$$d^2x/dt^2 + 2\beta dx/dt + \omega_0^2 x = A\cos(\omega t).$$

   Determine the particular solution of this equation.

6. (15%) Three stars with the same mass M form an equilateral triangle that rotates around the triangle's center as the stars move in a common circle around that center. The triangle has edge length L, what is the speed of the stars?

7. (15%) At the origin $x = y = 0$ a machine can shoot a ball with constant speed v_0 at any angle. Find the boundary of the region which can be reached by the ball.

Quantum Mechanics Qualification
Fall, 2015.

1. (5%) In what situation it is correct to directly solve the following equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right)\psi(\vec{r}) = E\,\psi(\vec{r})\,?$$

   Give a counter example.

2. (10%) Use some of the spherical harmonics of $Y_{l=1}^{m}(\theta,\phi)$, to construct your own normalized wave function which is an eigen-state of $\hat{L}_y$.

3. (7+8%) (a) In a 3-dim Hilbert space, construct two physical operators A and B (as two 3×3 matrices) and they satisfy: (1) A, B share one and only one common eigenvalue, also (2) $[A, B] \neq 0$. (b) Use your A, B to verify the generalized uncertainty principle.

4. (7+5+8%) Consider a perturbation $H' = \alpha x^4$ to the harmonic oscillator problem.
   (a) Work out the first-order correction to the eigen-energies of state $|n^{(0)}\rangle$. (hint: $a = (ip + m\omega x)/\sqrt{2\hbar m\omega}$ where ω is the natural frequency.)
   (b) No matter how small α is, the perturbation expansion will break down for some large enough n. Why?
   (c) If we make the perturbation time-dependent, $H' = \alpha x^4 e^{-t^2/\tau^2}$, between $t = -\infty$ and $t = +\infty$. What is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = +\infty$?

   You might find the following identity useful.

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\,\exp\frac{b^2}{4a}$$

5. (10%) Consider a central scattering potential $V(r) = V_0$ for $r < a$ and $V(r) = 0$ elsewhere, where V_0 is a constant. Use the Born approximation to evaluate the total scattering cross section in the limit of $(ka \ll 1)$, where k is the momentum of the incident particle.

6. (8+8%) (a) Write down the spatial and spinal wave function of the first excited state for two noninteracting electrons in an infinite potential well, $V(x) = 0$ for $-L \leq x \leq L$ and $V(x) = \infty$ elsewhere.
   (c) Similarly, write down the spatial wave function of the first excited state for two noninteracting spin-0 particles in the same potential well.

7. (7+7%) (a) Give an explicit example of two states in the Hydrogen atom with all quantum numbers specified and the $E1$ transition between the two states is allowed.
   (b) As in (a), provide an example where $E1$ transition is NOT allowed.

8. (10%) For a point particle, show that $\frac{\partial}{\partial x}$ is NOT a Hermitian operator in the position representation. (Hint: consider a general matrix element $\int \psi_1^*(x)\frac{\partial}{\partial x}\psi_2(x)dx$.)