Qualifying Exam: **Electrodynamics** March 8, 2015

1. (30%) Explain the following terms qualitatively and quantitatively.

(a) Group velocity and phase velocity (5%)

(b) Lorentz gauge and Coulomb gauge (5%)

(c) Maxwell stress tensor (5%)

(d) Kramers-Kronig relations (5%)

(e) Retarded Green function (5%)

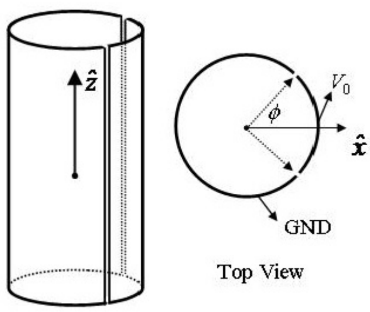
(f) Perfect conductor and super conductor (5%)

2. (10%, 10%) Green function

(a) What are Green's first identity and Green's theorem?

(b) For a point charge q **inside** a grounded conducting spherical shell of radius a, find the Green function $G(\mathbf{x},\mathbf{x}')$ that satisfies Dirichlet boundary condition. [Hint: the method of images.]

3. (10%, 10%) A very long hollow conducting tube with radius R is cut into two parts. The right part, which is one-fourth of the whole tube ($\phi = -\frac{\pi}{4}$ to $\frac{\pi}{4}$), is kept at potential V_0 and the left part is kept at ground ($V = 0$).



(a) Find the electric potential $\phi(r, \phi)$ inside the tube.

(b) Calculate the surface charge density $\sigma(r = R, \phi)$ on both parts of the tube and determine the capacitance per unit length. The solution of Laplace equation in cylindrical coordinate with z-symmetry is:

$$V(r,\ \phi) = \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \sin n\phi + D_n \cos n\phi\right).$$

[Hint: $\int_0^{\pi} \sin n\phi \cdot \sin m\phi \, d\phi = \int_0^{\pi} \cos n\phi \cdot \cos m\phi \, d\phi = \begin{cases} 0 & \text{if } n \neq m \\ \frac{\pi}{2} & \text{if } n = m \end{cases}$

4. (10%) If $\mathbf{E}$ and $\mathbf{B}$ are perpendicular in the laboratory and $|E| = 2|B|$, find a reference frame such that the field is pure electric or magnetic? If yes, what is the velocity of the reference frame relative to the laboratory? (10%)

[Hint: Gaussian unit $\mathbf{E}'_{||} = \mathbf{E}_{||}$, $\mathbf{E}'_{\perp} = \gamma_0\left(\mathbf{E}_{\perp} + \frac{\mathbf{v}_0}{c} \times \mathbf{B}_{\perp}\right)$ and $\mathbf{B}'_{||} = \mathbf{B}_{||}$, $\mathbf{B}'_{\perp} = \gamma_0\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}_0}{c} \times \mathbf{E}_{\perp}\right)$]

5.(10%, 10%)

(a) Starting from the Maxwell equations, derive the dispersion relation (i.e. the relation between the wave frequency ω and the propagation constant k) for a plane electromagnetic wave in an infinite and uniform medium of conductivity σ, electrical permittivity ε, and magnetic permeability μ.

(b) Assume that the medium is a good conductor, derive an expression for its skin depth δ.
[vector formula: $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$]

Qualification Exam for PhD Candidates (Classical Mechanics, March 2015)

1. (10%) Write down (without proof) the general solution of the wave equation
$$\frac{\partial^2 F(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F(x,t)}{\partial t^2} = 0$$

2. (15%) A particle with mass M decays at rest into a photon and another particle with mass m. Use the theory of relativity to calculate the total energy (including the rest energy) of the particle with mass m.

3. (15%) Consider the elastic scattering of two particles with masses M and m where $M > m$. The speed of M is V and the particle m is at rest. Find the maximum scattering angle θ of M after scattering.

4. (15%) Consider two coupled oscillators with the same mass m and spring constant k. The potential energy is
$$V = k(x_1^2 + x_2^2)/2 + \epsilon x_1 x_2$$
where ϵ is the coupling constant and x_i are the coordinates of the oscillators. Find the angular frequencies of vibration.

5. (15%) Consider the motion of a particle with mass m in a central force field and use the spherical polar coordinates (r, θ, ϕ). The potential energy is $V(r)$. (a) Write down the Lagrangian of the system. (b) Derive the Hamiltonian from the Lagrangian. (c) Write down the Hamilton's equations. (Do not solve the equations).

6. (15%) A rocket is fired at 60^0 to the local vertical line with an initial speed $v_0 = \sqrt{GM/R}$ where M is the mass of earth and R is its radius. Find the maximum distance from the center of the earth. The effect of rotation of earth can be neglected.

7. (15%) A ball of radius r and mass m is placed at rest on the top of a hemisphere of radius R. The moment of inertia of the ball about an axis passing through the center is $I = (2/5)mr^2$. If the ball rolls down from the top without slipping, at what height the ball loses contact with the hemisphere?

Qualifying Examination – Statistical Mechanics

Spring, 2015

1. (Fundamental concepts in statistical mechanics, 20 points)
   Please explain in brief the following terminologies in Statistical Mechanics.
   (a) Ergodic theorem and the postulate of equal *a priori* probability
   (b) Bose-Einstein condensation and its criterion
   (c) Correlation function
   (d) Critical phenomenon and order parameter
   (e) Fluctuation-dissipation theorem
   (f) (Classical) Liouville's theorem

2. (Canonical ensemble for classical gas, 20 points)
   (a) For an ideal classical gas consisting of N particles of mass m in a container of volume V and at temperature T, please
      (i) Write down its Partition function. (Note to include the factor $\frac{1}{N!}$ to prevent the Gibbs paradox.)
      (ii) Calculate its corresponding Helmholtz free energy, F. (Given the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.)
      (iii) Derive the familiar state equation, $PV=Nk_{\mathrm{B}}T$, by $P = -\left.\frac{\partial F}{\partial V}\right|_{T,N}$.
      (iv) Calculate the entropy of this gas by $S = -\left.\frac{\partial F}{\partial T}\right|_{V,N}$. Take two such containers of different gas and open the partition to allow them to mix. What is the entropy change after the mixing?
   (b) To take into account that real particles (1) repel each other at close range, imagine each particle to be a hard sphere of volume v and (2) attract each other at long range, add a potential energy term[1] $-\alpha\left(\frac{N}{V}\right)^2 V$ to the free energy in (ii) of (a). Show that modifications (1) and (2) lead us to the Van der Waals equation: $\left[P + \alpha\left(\frac{N}{V}\right)^2\right](V - Nv) = Nk_B T$.

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[1] The parameter, α, describes the strength of pairwise attractive force between particles, while $\left(\frac{N}{V}\right)^2$ comes from the joint probability. Finally, the V factor comes from integrating the energy density $-\alpha\left(\frac{N}{V}\right)^2$ over the whole volume to obtain the potential energy.

3. (Random walk, 20 points)

(a) A drunkard is kicked out of the pub. Being fully intoxicated, he walks forward and backward with equal probability. According to the senior high-school math, the probability of finding him at position *n* after *N* steps is $P(n,N) = C_{N_{\rightarrow}}^{N} \left(\frac{1}{2}\right)^{N_{\rightarrow}} \left(\frac{1}{2}\right)^{N_{\leftarrow}}$ where $N_{\rightarrow}$ and $N_{\leftarrow}$ represent the number of steps forward and backward, respectively. Naturally,

$$N = N_{\rightarrow} + N_{\rightarrow} \text{ and } n = N_{\rightarrow} - N_{\rightarrow}. \qquad (1)$$

Assume $N \gg n \gg 1$ so that the Stirling formula can be used to approximate all large factorials: $\lim_{N \gg 1} N! \approx N \ln N - N$. Show that $P(n,N)$ can be reduced to the Gaussian distribution: $P(n,N) \sim \frac{1}{\sqrt{N}} exp\left(-\frac{n^2}{4N}\right)$.

(b) Tom Witten of the University of Chicago has a simple theory[2] for the ridge-length distribution on a crumpled paper. He argues that any ridge of length ℓ must result from a consecutive (and thus hierarchical) decimation of the first and longest ridge, which equals roughly the length *L* of paper. So

$$\ell = L \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdots \qquad (2)$$

where ℓ happens to fall in the shorter $\frac{1}{3}$ (or longer $\frac{2}{3}$) half of the previous ridge for $N_{\frac{1}{3}}$ (and $N_{\frac{2}{3}}$) number of times throughout a total of $N = N_{\frac{1}{3}} + N_{\frac{2}{3}}$ folds. Equation (2) can be re-expressed as $\ln \frac{R}{\ell} = N_{\frac{1}{3}} \ln \frac{3}{1} + N_{\frac{2}{3}} \ln \frac{3}{2}$ or

$$\ln \frac{R}{\ell} = \left(N_{\frac{1}{3}} + N_{\frac{2}{3}}\right) \frac{\ln \frac{9}{2}}{2} + \left(N_{\frac{1}{3}} - N_{\frac{2}{3}}\right) \frac{\ln 2}{2} \qquad (3)$$

While $\left(N_{\frac{1}{3}} - N_{\frac{2}{3}}\right)$ mimics $N_{\rightarrow} - N_{\rightarrow}$ of Eq.(1), the first term of Eq.(3) can be thought of as describing the presence of wind. You see, a wind is going to sway and carry the drunkard along its direction. Therefore, the eventual position *n* (which role is now played by $\ln \frac{R}{\ell}$) of the drunkard depends on the wind speed, $\frac{\ln \frac{9}{2}}{2}$, and the "time", $N = N_{\frac{1}{3}} + N_{\frac{2}{3}}$, he stays out in the wind. Please use the above information to modify the result of (a) to predict the distribution function for ridge length ℓ on a crumpled sheet.

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[2] T. A. Witten, Rev. Mod. Phys. **79**, 643 (2007).

4. (One-dimensional Ising model, 20 points)

The Hamiltonian of one-dimensional Ising model is

$$H = -J\sum_{i=1}^{N-1} S_i S_{i+1} - B\sum_{i=1}^{N} S_i$$

where $S_i = \pm 1$ and B is the external magnetic field. When the coupling constant J is positive/negative, the Hamiltonian favors parallel/antiparallel alignments for the spins.

(a) Please show that the magnetization, $M = \sum_{i=1}^{N} \langle S_i \rangle$ where $\langle \ \rangle$ denotes the statistical average, is always zero in the absence of B. In other words, there is no phase transition into a magnetic state at any temperature T.

(b) Find the magnetic susceptibility, $\chi \equiv \frac{dM}{dB}$, when B is much less than both J and $k_B T$.

5. (Debye and Einstein models, 20 points)

The Dulong-Petit law states that the specific heat C_V of a crystal is a constant (of temperature T), which equals $\frac{k_B}{2}$ multiplied by its degrees of freedom. So it came as a huge blow and challenge when experiments showed that C_V in fact varies with T. This puzzle helped to bring about the advent of QM. Among the various theories that attempted to amend this discrepancy, the model proposed by the great Einstein turned out to be a blunder….

(a) Einstein assumed that the phonon frequency ω is a constant. Please derive his prediction for the temperature dependence of C_V.

(b) In contrast, Debye allowed the frequency to vary as $\omega = vk$ where v is the sound velocity in the solid and k the wave momentum. Please again derive $C_V(T)$ based on the Debye model and compare with that of Einstein's at extremely low and high temperatures. (Note that the largest value of k is not zero, but equals $\frac{\pi}{a}$ where a denotes the lattice constant of the solid.)

6. (Fitting of real experimental data, 加分題, 10 points)

Cutting a long rod of length L by half, one gets two rods of length $L/2$. Further cut renders four rods of length $L/4$. Repeating this procedure by $n \gg 1$ times gives 2^n number of length-$L/2^n$ rods. Plot the number $N(\ell)$ as a function of rod length ℓ in a histogram（長條圖）for ALL rods that ever exist. What kind of distribution function do you obtain? Note that the answer is not $N(\ell) = L/\ell$. (註：長條圖的做法是先針對 x-軸參數選取一個固定間隔，並且把在每一間隔的棍子數目加起來當成新的 y 軸參數；由於棍子的長度越差越小，數據點在橫軸的分佈是 x 越小、點越密，因此小 x 的長條圖間隔會分到比較多的數據點，因此可以預期最後長條圖給出的 $N(\ell) \propto \frac{1}{\ell^{\alpha}}$分佈，對應的冪次方$\alpha$應該會大於 1）

Quantum Mechanics Qualification
Spring, 2015.

1. (7%) In what situation it is correct to directly solve the equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right)\psi(\vec{r}) = E\,\psi(\vec{r})\,?$$

   Give a counter example.

2. (8%) Give two physical examples that one of the associated Hilbert spaces is finite dimensional and the other one is $\infty-$dimensional.

3. (10%) Use some of the spherical harmonics of $l = 2$ to construct a normalized wave function which is an eigen-state of $\hat{L}_x$.

4. (7+8%) (a) In a 3-dim Hilbert space, construct two physical operators A and B and they satisfy: (1) A, B share one and only one common eigenvalue, also (2) $[A, B] \neq 0$. (b) Use your A, B to verify the generalized uncertainty principle.

5. (10%) Work out all the Clebsch-Gordan coefficients of $\frac{1}{2} \times \frac{1}{2}$.

6. (5+5+5+5%) An electron is trapped in a 2-dimensional infinite potential well in a rectangular area $a \leq x \leq a{+}L$ and $b \leq y \leq b{+}2L$. (a) Write down the corresponding Schrodinger wave function with proper normalization. (b) What is the probability of finding the ground state electron in the rectangular region $a \leq x \leq a + L/3$ and $b \leq y \leq b + L$?
   If 10 more electrons are filled in and we assume there is no interaction among the electrons,
   (c) what is the ground state energy of the 11-electron system? (d) What's the minimal energy to excite the system?

7. (7+5+8%) Consider a perturbation $H' = \alpha x^4$ to the harmonic oscillator problem.
   (a) Show that the first-order correction to the unperturbed eigen-energies are

$$E_n^{(1)} = \frac{3\hbar^2\alpha}{4m\omega^2}[1 + 2n + 2n^2]$$

   (b) No matter how small α is, the perturbation expansion will break down for some large enough n. Why?
   (c) If we make the perturbation time-dependent, $H' = \alpha x^4 e^{-t^2/\tau^2}$, between $t = -\infty$ and $t = +\infty$. What is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = +\infty$?

You might find the following identity useful.

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\,\exp\frac{b^2}{4a}$$

8. (10%) Consider the S-wave neutron-neutron scattering where the interaction potential is approximated by $V(r) = V_0 \vec{S}_1 \cdot \vec{S}_2 e^{-r/a}$, where $\vec{S}_1$ and $\vec{S}_2$ are the spin vector operators of the two neutrons, and $V_0 > 0$. Find the differential cross section in the first Born approximation.