Quantum Mechanics Qualification
Fall, 2014.

1. (5% each)Briefly answer the following questions:

   - What is Hilbert space?
   - What is the value of $\hbar$ (and specify the unit clearly)? And give one example how to experimentally determine $\hbar$.
   - In a spherically symmetrical potential, a state is described by a normalized wave function $\psi(r,\theta,\phi) = R(r)\times\frac{1}{\sqrt{2}}[\frac{3}{5}Y_3^2 - i\frac{4}{5}Y_3^1 + iY_3^{-1}]$. What is $\langle\psi|L_x^2 + L_y^2|\psi\rangle$?
   - What is the Hermitian conjugate of the operator $x\frac{\partial}{\partial x} - iy\frac{\partial}{\partial z}$?
   - What are the Clebsch-Gordan coefficients?
   - How the magnetic resonance imaging (MRI) works?
   - What is the Aharonov-Bohm effect?
   - What is Bell's inequality?

2. (15%) Two particles of the same mass m are confined to move in one dimension and they are connected by a spring with spring constant k. Suppose that the total momentum of the system is p, find all possible total energies for the following cases: (i) two particles are different (ii) two particles are identical fermions (iii) two particles are identical bosons.

3. (8%+7%) A point particle of mass m is subject to the following central potential

$$V(r) = \begin{cases} \infty, & r < r_1 \\ -\frac{\hbar^2}{m}\frac{1}{r^2}, & r_1 < r < r_1 + a \\ V_0(>0), & r_1 + a < r < r_1 + a + \triangle \\ 0, & r_1 + a + \triangle < r\,. \end{cases}$$

   (a) In the limit of $V_0 \to \infty$, find the corresponding eigen-energies and the normalized wavefunctions for $l = 1$ bound state. Recall that the Laplacian operator is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

   in the spherical coordinate.
   (b) Now consider the case that V_0 is finite but much larger than the ground state energy obtained in (a). If initially the particle was in the first excited state, what properties (e.g. energy, wave function, angular momentum, and etc) can you say about the particle which tunnels through the V_0 barrier and escapes?

4. (15 %) A particle is initially in its ground state in a one-dimensional harmonic oscillator potential. At $t = 0$, a perturbation $V(x,t) = V_0 x^3 e^{-t/\tau}$ is turned on. Calculate to first order the probability that, after a sufficiently long time ($t \gg \tau$), the system will have made a transition to a given excited state; consider all final states.

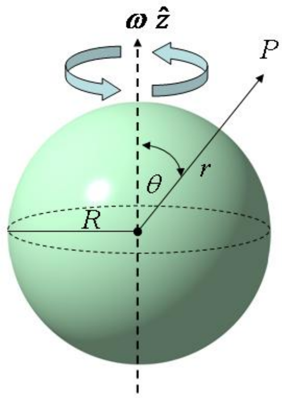
5. (15%) Consider the S-wave neutron-neutron scattering where the interaction potential is approximated by $V(r) = V_0 \vec{S}_1 \cdot \vec{S}_2 e^{-r/a}$, where $\vec{S}_1$ and $\vec{S}_2$ are the spin vector operators of the two neutrons, and $V_0 > 0$. Find the differential cross section in the first Born approximation.

Ph.D. Qualify Exam **Classical Electrodynamics** Fall 2014

1. (20 %) Explain the following items:
   (a) Linear and circular polarization of light
   (b) Kramers-Kronig relations
   (c) Cherenkov radiation
   (d) Synchrotron radiation
   (e) Skin depth
   (f) Pseudovector (axial vector)

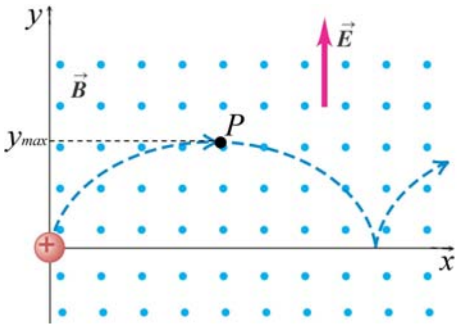
2. (20 %) [Maxwell equations and EM waves]
   (a) Write down the differential form of the four Maxwell equations.
   (b) Derive the wave equations in free space for the electric field from Maxwell equations. You may need this identity $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$.
   (c) Show that the electric field and magnetic field are perpendicular to each other in an electromagnetic wave.
   (d) A laser beam at 532 nm has an intensity of 1 mW/mm^2. Find the root-mean-square value of its electric field and magnetic field. ($\varepsilon_0 = 8.85\times10^{-12}$ C^2/N·m^2, $\mu_0 = 4\pi\times10^{-7}$ N/A^2)

3. (20 %) [Magnetostatics] A hollow spherical shell of radius R, carrying uniform surface charge density σ is spinning at constant angular velocity ω as shown in the figure. (a) What is the surface current density $\vec{K}$ (current per unit length) at a point on the shell with coordinate (R, θ, ϕ)? (Notice: $\vec{K}$ is a vector, and your answer must include the direction) (b) Find the magnetic field $\vec{B}$ at point P whose coordinate is (r, θ, ϕ). Express your answer for both P inside and outside the shell. (hint: you may find the vector potential $\vec{A}$ or directly use Biot-Savart law as $B(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\vec{K}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}da'$.



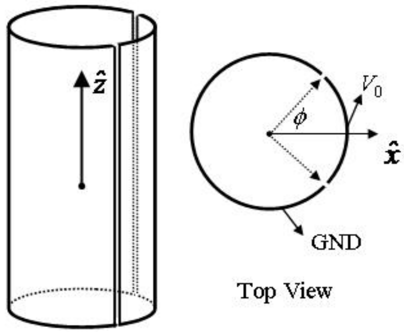
However, since $\nabla\times\vec{B} = \mu_0\vec{J} = 0$ at point P where no current is present, we can define a scalar potential Ψ such that $\vec{B} = \nabla\Psi$, and $\nabla^2\Psi = 0$. Due to ϕ symmetry, you can take the result from separation of variables and the solution of the Laplace equation in this case is $\Psi = \left[A_1 r + \dfrac{B_1}{r^2}\right]\cos\theta$. Then apply appropriate boundary conditions. You may need the gradient operator in spherical coordinate: $\nabla f = \dfrac{\partial f}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial f}{\partial\theta}\hat{\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial\phi}\hat{\phi}$.

4. (15 %) [Force on a charged particle] A particle with mass m and charge $+q$ starts from rest at the origin under a uniform electric field $\vec{E} = E\,\hat{j}$ and a uniform magnetic field $\vec{B} = B\,\hat{k}$. Neglecting earth's gravity, the particle is accelerated to the $+y$-direction by the E field, but the magnetic force bends it to the right and the resulting trajectory is shown in the figure. At point P, the particle reaches its maximum height called $y_{\max}$. Determine the curve of the trajectory and express $y_{\max}$ in terms of m, q, E, B.



5. (25 %) [Electrostatics]

A very long hollow conducting tube with radius R is cut into two parts. The right part, which is one-fourth of the whole tube ($\phi = -\frac{\pi}{4}$ to $\frac{\pi}{4}$), is kept at potential V_0 and the left part is kept at ground ($V = 0$). (a) Find the electric potential $V(r,\phi)$ inside the tube. (b) Calculate the surface charge density $\sigma(R,\phi)$ on both parts of the tube and hence determine the capacitance per unit length. The solution of Laplace equation in cylindrical coordinate with z-symmetry is:



$V(r,\phi) = \sum_{n=1}^{\infty}(A_n r^n + B_n r^{-n})(C_n \cos n\phi + D_n \sin n\phi)$. You may also need the orthonormal condition for Fourier series:

$$\int_0^{\pi} \sin n\phi \cdot \sin m\phi \cdot d\phi = \int_0^{\pi} \cos n\phi \cdot \cos m\phi \cdot d\phi = \begin{cases} 0, & if\ n \neq m \\ \frac{\pi}{2}, & if\ n = m \end{cases}.$$

Qualifying Examination – Statistical Mechanics

Spring, 2014

1. (Fundamental concepts in statistical mechanics, 30 points)
   Please EXPLAIN (not just describe or write equations/formula) the following terminologies in Statistical Mechanics.
   (a) Pure state and density matrix
   (b) Bose-Einstein Condensation and superfluidity
   (c) Pauli paramagnetism and Landau dia-magnetism

2. (Harmonic oscillators, 30 points)
   Consider an idea gas absorbed on a 2D surface at temperature T. The energy of the ith molecule is

   $$E_i = a(x - x_i)^2 + b(y - y_i)^2 + (p_x^2 + p_y^2)/2m$$

   where a and b are constants and x_i and y_i are position of that molecule.
   (a) Consider these molecules to be a classical gas. Determine the average energy of such molecule.
   (b) Calculate the uncertainty of the position and momentum of each molecule.
   (b) Consider them to be a quantum gas (bosons), what will be the difference (or similarity) for above two results ? why ?

3. (Free bosons, 20 points)
   Consider now a number conserved Bose gas with energy dispersion $\varepsilon_{\mathrm{p}} = \mathrm{C}|\mathbf{p}|^{a}$:
   (a) show that there will be a Bose-Einstein condensation if $d>a$ (b) Show that the critical temperature scales with the total number of particles, N, as $T_c \propto N^{a/d}$.
   (Hint: you do not need to do the full calculation, and may use dimensional argument for the density of states first).

4. (Free fermions, 20 points)
   Consider a system of spin polarized fermions in 2D space : (a) Write down the expression of the total number of particles, N, and total energy, U, at zero temperature and zero magnetic field. (b) What is the eigenstate energy and wave function if now a magnetic field (B) is applied in the z axis ? (c) What is Landau level degeneracy. (d) Calculate the magnetization, M, and its susceptibility μ in the zero B limit at zero temperature.

Qualification Exam [Classical Mechanics]

1. (20 %) Consider a system of two coupled oscillators with the same mass m and spring constant k. The potential energy is

$$V(x_1, x_2) = \frac{k}{2}\left(x_1^2 + x_2^2\right) + \epsilon x_1 x_2$$

where ϵ is the coupling constant and x_1, x_2 are the coordinates of the two oscillators. Find the eigenfrequencies of vibration.

2. (20 %) Find Hamilton's equations of motion for an unharmonic oscillator system for a particle of mass m and the potential energy is given by

$$V(x) = \frac{k}{2}x^2 + \frac{a}{4}x^4$$

where k and a are constants. (You don't have to solve these equations)

3. (20 %) A satellite of mass m was orbiting around the earth in a stable circular orbit with a velocity of v. It was then hit by a space debris vertically with an impulse of Δp. Find the motion of the satellite after the incident.

4. (20 %) A charge particle with mass m enters into a static electro-magnetic field with a velocity v in the x-direction. The electric field is in the y-direction and the magnetic field is in the z-direction: $\mathbf{E} = (0, E, 0)$, $\mathbf{B} = (0, 0, B)$. Find the motion of the charge particle.

5. (20 %) A proton hits a proton at rest, producing antiprotons:

$$p + p \;\rightarrow\; p + p + p + \bar{p}$$

Calculate the minimum energy of th incident proton. The proton's rest mass is 1 GeV/c^2.