Qualifying Examination – Statistical Mechanics

Spring, 2014

1. (Fundamental concepts in statistical mechanics, 30 points)
   Please explain in brief the following terminologies in Statistical Mechanics.
   (a) Canonical and grand canonical ensemble
   (b) Gibbs' paradox
   (c) Bose-Einstein condensation
   (d) Landau level
   (e) Pauli paramagnetism

2. (Einstein's theory of the heat capacity of a solid, 10 points)
   Describe Einstein's theory of the heat capacity of a solid, and explain why his theory disagrees with the experimental data for real crystals at very low temperature.

3. (micro-canonical, canonical, and grand-canonical ensembles, 10 points)
   Lagrange multipliers allow one to find the extremum of a function $f(x)$ given a constraint $g(x)=g_0$. One sets to zero the derivative of $f(x)+\lambda(g(x)-g_0)$ with respect to x. Let us now use Lagrange multipliers to find the maximum of the entropy $S=-k_B\sum p_i \ln p_i$ by constraining

   (a) the normalization $\left(\sum p_i = 1\right)$,

   (b) the energy $\left(\langle E\rangle = \sum E_i p_i\right)$, and

   (c) the number $\left(\langle N\rangle = \sum N_i p_i\right)$

   to obtain the distribution function p_i for micro-canonical, canonical, and grand-canonical ensembles, respectively.

4. (classical molecular gas, 10 points)
   Consider a classical ideal gas at temperature T. The energy of a molecule is $\varepsilon = ax^2 + bx + \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right)$ where a and b are constants. Determine the average energy of a molecule.

5. (quantum harmonic oscillator, 10 points)

A one-dimensional quantum harmonic oscillator, whose ground-state energy is $\hbar w/2$, is in equilibrium with a heat reservoir at temperature T. Determine the mean value of the oscillator's energy.

6. (entropy for an ideal classical gas, 10 points)
   (a) In a model of ideal classical gas, all particles exhibit the same speed v in random directions. Calculate the average value of the relative speed.
   (b) If this gas is stored inside an adiabatic (or thermally insulated) container of volume V, how much entropy can it gain by reaching equilibrium and obeying the Boltzmann distribution for the particle energy?

7. (black hole thermodynamics, 20 points)
   Astrophysicists have long studied black holes: the end state of massive stars which are too heavy to support themselves under gravity. As the matter continues to fall into the center, eventually the escape velocity reaches the speed of light. After this point, the in-falling matter cannot ever communicate information back to the outside. A black hole of mass M has radius $R_S = G\dfrac{2M}{c^2}$ where the gravitational constant $G = 6.67\times10^{-8}\,\mathrm{cm}^3/\mathrm{g\,s}^2$ and c is the speed of light. By combining methods from quantum mechanics and general relativity, Hawking calculated the emission of radiation from a black hole and found it to follow the perfect black-body radiation at a temperature $T_{\mathrm{bh}} = \dfrac{\hbar c^3}{8\pi G M k_B}$. According to Einstein's theory, the energy of the black hole is $E = Mc^2$.
   (a) Calculate the specific heat of the black hole.
   (b) Calculate the entropy of the black hole, by using the definition of temperature $1/T = \partial S/\partial E$ and assuming the entropy is zero at mass $M = 0$.

Qualification Exam (Classical Mechanics)

Feb 2014

1. (15) A block is projected up an incline at angle θ. The block returns to its initial position with half its initial speed. Find the coefficient of kinetic friction.

2. (15) A photon γ hits an electron at rest, producing an electron-position pair:

$$\gamma + e^- \to e^+ + e^- + e^-.$$

Calculate the minimum energy of the incident photon. The electron's rest mass is 0.5 MeV/c^2.

3. (10) What is the total cross section for the elastic scattering of a beam of particles of radius r from a fixed solid sphere with radius R?

4. (15) A particle with mass m and total energy E moves in one dimension. The potential energy is $V(x) = C|x|$ where C is a positive constant. Using action-angle variables to determine the period of the motion.

5. (15) Consider the motion of a particle with mass m in a central force field and use the spherical polar coordinates (r, θ, ϕ). The potential energy is $V(r)$. (a) Write down the Lagrangian of the system. (b) Derive the Hamiltonian from the Lagrangian. (c) Write down the Hamilton's equations.

6. (15) A uniform ladder leans against a smooth vertical wall. The initial angle between the floor and the ladder is θ_0. If the floor is also smooth, then the

ladder will slide down. Find the angle between the floor and the ladder when the ladder loses contact with the vertical wall.

7. (15) A bucket of water is set spinning about its symmetry axis with constant angular velocity ω. Determine the shape of the water in the bucket.

Quantum Mechanics Qualification
Spring, 2014.

1. (5% each)Briefly answer the following questions:

   - For a given state $|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|2\rangle + \frac{i}{\sqrt{3}}|3\rangle$, where $|1\rangle, |2\rangle, |3\rangle$ are three orthonormal basis. What is the probability of finding $|\psi\rangle$ in $|3\rangle$?
   - Continue from the previous question, if there is another state $|\phi\rangle = |1\rangle + \frac{i}{\sqrt{2}}|2\rangle - \sqrt{3}|3\rangle$, $\langle 2|\psi\rangle\langle\phi|1\rangle =$?
   - In a spherically symmetrical potential, a state is described by a normalized wave function $\psi(r,\theta,\phi) = R(r) \times [\sqrt{2/7}Y_4^2 - i\sqrt{1/7}Y_4^0 + i\sqrt{4/7}Y_4^{-1}]$. What is $\langle\psi|L_x^2 + L_y^2|\psi\rangle$?
   - What is the Hermitian conjugate of the operator $x\frac{\partial}{\partial x}$?
   - What are the Clebsch-Gordan coefficients?
   - What is Path Integrals?
   - What is spontaneous emission?

2. (15%) Consider two particles of the same mass m in one dimension and they are connected by a spring with spring constant k. Suppose that the total momentum of the system is p, find all possible total energies for the following cases: (i) two particles are different (ii) two particles are identical fermions (iii) two particles are identical bosons.

3. (8%+7%) A point particle of mass m is subject to the following central potential

$$V(r) = \begin{cases} \infty, & r < a \\ 0, & a < r < a + b \\ V_0(>0), & a + b < r < a + b + \triangle \\ 0, & a + b + \triangle < r\,. \end{cases}$$

   (a) In the limit of $V_0 \to \infty$, find the corresponding eigen-energies and the normalized wavefunctions for $l = 0$ bound state. Recall that the Laplacian operator is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

   in the spherical coordinate. (b) Now consider the case that V_0 is finite but much larger than the ground state energy obtained in (a). Use the WKB approximation for tunneling to estimate the escaping rate (= probability per unit time) of the particle in the lowest bound state.

4. (10%) A point particle of mass m and incident energy E is scattering off the potential $V(\vec{r}) = ge^{-r^2/R^2}$. Calculate the first Born approximation to the differential and total cross sections.

5. (10%+5%+10%) Consider a perturbation $H' = \alpha x^4$ to the harmonic oscillator problem.

(a) Show that the first-order correction to the unperturbed eigen-energies are

$$E_n^{(1)} = \frac{3\hbar^2\alpha}{4m\omega^2}[1 + 2n + 2n^2]$$

(b) No matter how small α is, the perturbation expansion will break down for some large enough n. Why?

(c) If we make the perturbation time-dependent, $H' = \alpha x^4 e^{-t^2/\tau^2}$, between $t = -\infty$ and $t = +\infty$. What is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = +\infty$?

You might find the following identity useful.

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} \exp\frac{b^2}{4a}$$

Ph.D. Qualify Exam **Classical Electrodynamics** Spring 2014

1. (20 %) [Maxwell equations and EM waves]
   (a) Write down the differential form of the four Maxwell equations.
   (b) Derive the wave equations in free space for the electric field from Maxwell equations. You may need this identity $\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$.
   (c) Show that the electric field and magnetic field are perpendicular to each other in an EM wave.
   (d) A laser beam has an intensity of $1\mathrm{W/m^2}$. Find the peak value of its electric field and magnetic field. ($\mu_0 = 4\pi \times 10^{-7}$ N/A^2)

2. (20 %) Explain the following items:
   (a) Linear and circular polarization of light.
   (b) Phase velocity and group velocity
   (c) Multipole expansion of electric potential.
   (d) Rayleigh scattering.
   (e) Liénard-Wiechert potential.
   (f) Kramers-Kronig relations.

3. (20 %) [Green function]

   To solve an electrostatic problem, $\nabla^2\Phi(\vec{r})=-\rho(\vec{r})/\varepsilon_0$, for arbitrary charge distribution $\rho(\vec{r})$, one may use the Green function approach. The Green function for 3-dimensional Poisson equation satisfies:

   $$\nabla^2 G(\vec{r},\vec{r}\,')=\delta(\vec{r}-\vec{r}\,') \text{ ----- (1)}$$

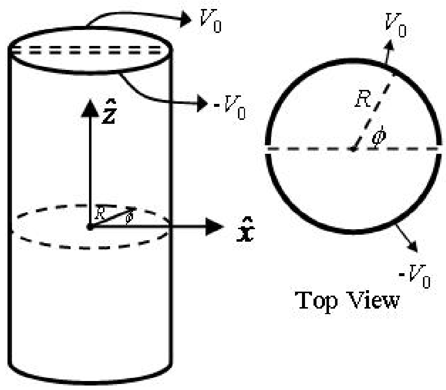
   (a) Brief explain how we can obtain $\Phi(\vec{r})$ by solving $G(\vec{r},\vec{r}\,')$.
   (b) To solve $G(\vec{r}-\vec{r}\,')$, first find the Fourier transform of the Green function, i.e. find $\tilde{G}(\vec{k})=\frac{1}{(2\pi)^{3/2}}\int G(\vec{r})e^{-i\vec{k}\cdot\vec{r}}d^3\vec{r}$, by taking the Fourier transform of equation (1).
   (c) Find the Green function $G(\vec{r}-\vec{r}\,')$ by making an inverse transform of $\tilde{G}(\vec{k})$, i.e. find $G(\vec{r})=\frac{1}{(2\pi)^{3/2}}\int \tilde{G}(\vec{k})e^{i\vec{k}\cdot\vec{r}}d^3\vec{k}$. Show this is exactly the form of Coulomb potential. (You may need this identity $\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$.)
   (d) For a three-dimensional Helmholtz equation, the Green's function satisfies:

   $$\left(\nabla^2-\lambda^2\right)G(\vec{r}-\vec{r}\,')=\delta(\vec{r}-\vec{r}\,') \text{ ------ (2)}$$

   where λ is a constant. Repeat the same procedure as in (b) and (c), find $G(\vec{r}-\vec{r}\,')$ and obtain the so-called Yukawa potential. (You may need this identity $\int_0^\infty \frac{x^2}{x^2+a^2}\frac{\sin x}{x}dx=\frac{\pi}{2}e^{-a}$.)

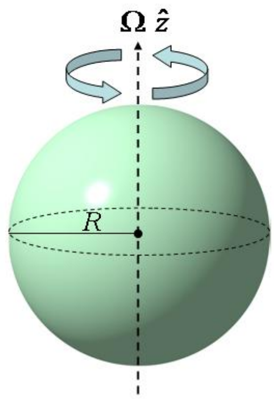
4. (20 %) [Electrostatics]

A very long hollow conducting tube with radius R is cut into halves through its center. The upper half is kept at potential V_0 and the lower half is kept at $-V_0$. Find the electric potential inside and outside the tube. The general solution of Laplace equation in cylindrical coordinate with z-symmetry is $\Phi(r,\phi) = \sum_{n=1}^{\infty}(A_n r^n + B_n r^{-n})(C_n \cos n\phi + D_n \sin n\phi)$. You may need this identity: $\int_0^{\pi} \sin n\phi \cdot \sin m\phi \cdot d\phi = \begin{cases} 0, & if\ n \neq m \\ \frac{\pi}{2}, & if\ n = m \end{cases}$.



5. (20 %) [Magnetostatics]

An insulating solid sphere of radius R has a spherically symmetric charge distribution $\rho(r)$. The sphere is rotating about an axis through its center at angular frequency Ω.



(a) Give an expression for the electric potential function $\Phi(\vec{r})$ at any point $\vec{r}$ in terms of an integral involving $\rho(r)$. The potential is zero at infinity.

(b) Give an expression for the current density $\vec{J}(\vec{r})$ in terms of the quantities given at any point inside the sphere.

(c) Give an expression for the magnetic field $\vec{B}(\vec{r})$ at any point $\vec{r}$ in terms of an integral involving the current density you find in (b).

(d) Find a simple expression for the ratio $\dfrac{|B(0)|}{\Phi(0)}$ in terms of only Ω and some constants.

(e) What is the direction of $\vec{B}(0)$ if $\rho(r)$ is positive everywhere?

You may need this identity: $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\cdot\vec{B} - (\vec{A}\cdot\vec{B})\cdot\vec{C}$