Qualifying Examination [Classical Mechanics]

9/28/2013

1. (20%) A pointlike mass m is undergoing a three-dimensional projectile motion in a uniform gravitational field g. Consider that the air resistance is negligible.

- (a) Write the Hamiltonian function for this problem $H = H(p_i, q_i)$, choosing as generalized coordinates the Cartesian coordinates (x, y, z) with the z axis pointing in the vertical direction.
- (b) Show that these equations lead to the known equations of motion for Projectile motion and that the Hamiltonian function corresponds to the Total Mechanical energy of the particle.
- (c) Once the particle reaches the highest point in its trajectory, a retarding force proportional to the velocity of the particle starts acting. Assume the proportionality constant to be known and given by k. Calculate the vertical velocity as a function of time for the descending particle, and find the terminal velocity.

2. (20%) A rigid body consists of six particles, each of mass m, fixed to the ends of three light rods of length $2a$, $2b$, and $2c$ respectively, the rods being held mutually perpendicular to one another at their midpoints.

- (a) Write down the inertia tensor for the system in the coordinate axes defined by the rods;
- (b) Find angular momentum and the kinetic energy of the system when it is rotating with an angular velocity ω about an axis passing through the origin and the point (a, b, c).

3. (20%) Many features of the orbits around a nonrotating black hole can be well approximated within the Newtonian equations of motion via a pseudo-Newtonian gravitational potential for $r > r_*$

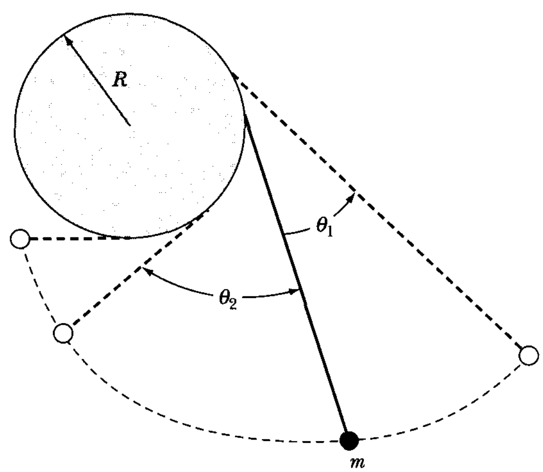
$$\Psi = -\frac{GM}{(r - r_*)}.$$

Consider only the region outside the black hole "horizon" $r_* = 2GM/c^2$, with M the mass of the "black hole" and $m\,(\ll M)$ the mass of the orbiting test particle.

- (a) Calculate the total energy and obtain the effective potential $\Psi_e(\tilde{l}, r)$ defined via the conserved total energy
$$E = m\left[\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \Psi_e\right],$$
where the conserved angular momentum $l = m\tilde{l}$. Sketch the (dimensionless) function Ψ_e/c^2, for a few values of $\alpha \equiv c\tilde{l}/(GM)$ in the range 0-5, using the dimensionless coordinate $x = r/r_* = rc^2/(2GM)$.
- (b) Over what range of r do circular orbits exits? For such a circular orbit, what is the corresponding angular velocity $\Omega(r)$?
- (c) Over what range of r are the circular orbits unstable?
- (d) For what range of α will a particle dropped nearly from rest very far away ($E = 0$) be swallowed by the black hole?

4. (20%) A pendulum is constructed by attaching a mass m to an extensionless string of length l. The fixed end of the string is connected to a fixed vertical disk of radius $R\,(R < l/\pi)$, as shown in the figure below. The motion of the pendulum occurs in the plane of the figure and the string connection to the disk is at a place such that in the oscillations of interest here the string is always tangent to the disk.

- (a) Find the Lagrangian and the equation of the motion for the pendulum.
- (b) Expand θ about some angle θ_0 and assume that $\epsilon \equiv \theta - \theta_0$ is small. Calculate the period of the small oscillations.
- (c) In the small oscillations regimes find the line about which the angular motion extends equally in either direction (i.e. $\theta_1 = \theta_2$).



5. (20%) Use following information to answer these questions: The rest-mass energy of an electron and an charged pion are 500 keV and 140 MeV respectively. The average lifetime of a charged pion at rest is 2.60×10^{-8}.

- (a) Consider a liner accelerator in which electrons are accelerated to 50 GeV per electron. The accelerator is 3 km long. The electrons are accelerated in "bunches" that is 1 cm long as measured in the laboratory frame of reference. Calculate the apparent length of the accelerator and a bunch of electrons as measured in the reference frame of the 50-GeV electrons.
- (b) Suppose an electron with total energy of 0.5 GeV collides head-on with a positron with total energy of 0.5 GeV. The electron and position annihilate and create a pair of back-to-back charged pions:
$$e^+ + e^- \to \pi^+ + \pi^-.$$
What is the average lifetime of these pions, as measured in the laboratory reference frame?
- (c) Suppose an electron with total energy of 0.5 GeV collides head-on with a positron with total energy of 0.5 GeV. The electron and position annihilate and create a pair of back-to-back neutral pions:
$$e^+ + e^- \to \pi^0 + \pi^0.$$
Assume that the mass of the neutral pion is the same as that of the charged pion. By far the most common decay mode of the π^0 is to two photons: $\pi^0 \to \gamma + \gamma$. What is the minimum possible photon energy and the maximum possible photon energy measured in the laboratory frame of reference?

Qualifying Examination – Statistical Mechanics

Fall, 2013

1. (some fundamental concepts in statistical mechanics, 30 points)
   Please explain in brief the following terminologies in Statistical Mechanics.
   (a) Definition of temperature in statistical mechanics
   (b) Debye model versus Einstein model
   (c) Fluctuation-dissipation theorem
   (d) Liouville's theorem
   (e) Pauli paramagnetism

2. (ideal Fermi gas in 3-D, 30 points)
   (a) Considering an ideal Fermi gas of spin S=5/2 at zero temperature, calculate the total internal energy U by summing the electron energy all the way up to the Fermi energy ε_{F}
   (b) Calculate the Fermi Pressure, P
   (c) Derive the Pauli magnetic susceptibility in the limit of zero magnetic field.

3. (classical gas and negative temperature, 20 points)
   Consider a collection of N particles sitting on some crystal lattice in which the energy of each particle can assume only two distinct values, 0 and E $(E > 0)$. n_0, n_1 are the occupation numbers of these two energy levels, respectively.
   (a) Write down the partition function at a given temperature, T.
   (b) Find the values of n_0, n_1 when in equilibrium, and calculate their mean fluctuations.
   (c) Calculate the total energy, U, and the entropy, S.
   (d) Define the absolute temperature T as $\frac{1}{k_{\mathrm{B}}T} \equiv \frac{1}{k_{\mathrm{B}}} \frac{\partial S}{\partial U}$. Show that T can be negative, and explain when it may happen.

4. (Gibbs paradox and entropy, 20 points)
   (a) Calculate the free energy for an ideal classical gas of N identical particles of individual mass m, confined in a container of volume V and temperature T.
   (Hint: the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$)
   (b) Take two such containers of DIFFERENT gas and open the partition to allow them to mix. What is the entropy change after the mixing?
   (c) Had the two gases in (b) been the same, what is the entropy change due to the mixing? Explain what problem happens and how to resolve it.

Qualifying Examination - Quantum Mechanics

Fall, 2013

1. (important terminologies in quantum mechanics, 30 points)

   Please explain in brief the following terminologies in QM.

   (a) Aharonov-Bohm effect

   (b) Clebsch-Gordan coefficients

   (c) Hyperfine interaction

   (d) Selection rules (for angular momentum, for instance)

   (e) Spontaneous emission

   (f) WKB approximation

2. (Delta-function potential, 20 points)

   (a) Find the ground state energy for attractive delta-function potentials $V(x) = -\alpha \cdot \delta(x)$

   (b) Using above results, present an argument why there must be a bound state for a general 1D short-ranged attractive potential (not delta function).

   (c) Find the ground state energy for $V(x) = -\alpha \cdot \delta(x - L/2) - \alpha \cdot \delta(x + L/2)$.

   (d) Draw a schematic plot to show how the– ground-state energy changes with L, but there is no need to solve for its exact value except at the asymptotic cases of $L = 0$ and $L = \infty$.

3. (Spin, 15 points)

   An electron of spin $\left| S_x = \frac{1}{2} \right\rangle$ is injected with velocity $(v,0,0)$ into $x \geq 0$ regime where there is a magnetic field $(0,0,B)$ (while $B = 0$ elsewhere).

   (a) Assuming the electron follows a classical trajectory, what will be the radius of its trajectory ? What is the time, T, it will need to eventually exist the regime of finite magnetic field ?

   (b) Write down the Hamiltonian to describe the spin degree of freedom (quantum mechanically) in the co-moving frame.

   (c) What is its spin state after time, T ?

4. (time-dependent and –independent perturbation theories, 20 points)

   Consider a perturbation $H' = \alpha x^4$ to the harmonic oscillator problem.

   (a) Show that the first-order correction to the unperturbed eigen-energies are

   $$E_n^1 = \frac{3\hbar^2 \alpha}{4m^2 \omega^2}\left[1 + 2n + 2n^2\right]. \text{ (8 points)}$$

   (b) Argue that no matter how small α is, the perturbation expansion will break down for some large enough n. What is the physical reason? (4 points)

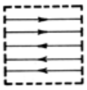
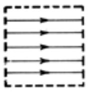
(c) If we are careful with how the perturbation is turned on; say, $H' = \alpha x^4 e^{-t^2/\tau^2}$ between $t = -\infty$ and $t = \infty$, what is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = \infty$? (8 points)

5. (Addition of angular momentum, 15 points)
Derive all spin eigenstates for two identical bosons of spin-1.

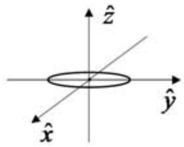
1. (25 %) [Maxwell equations and EM waves]

   (a) Write down the differential form of the four Maxwell equations.

   (b) The figures are static magnetic field lines. Which one(s) violate Maxwell's equation? Explain.

   (A)  (B)  (C)  (D)  (E) 

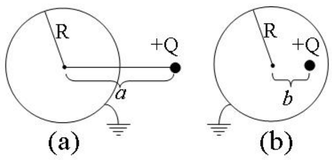
   (c) If magnetic monopoles exist, which equation(s) will be modified? And how?

   (d) A small current loop lying on the *x-y* plane is driven by a AC current $I = I_0 \cos \omega t$. Describe the angular distribution of the emitted power of the radiation.

   (e) The average intensity of the sunlight at Earth's surface is 1400 W/m^2. Estimate the magnitude of the magnetic field produced by the sunlight. $\mu_0 = 4\pi \times 10^{-7}$ N/A^2

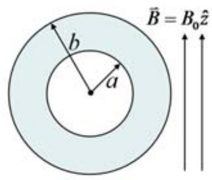
2. (25 %) [Electrostatics]

   (a) A point charge Q is placed outside a grounded conducting hollow sphere. Find the net force between the point charge and the sphere. (Hint: you may find an image charge $q' = -(RQ/a)$, placed a distance R^2/a to the right of the sphere center. But first you need to prove this arrangement can satisfy proper boundary conditions.)

   (b) Now the point charge is moved inside the sphere and is at a distance *b* from the center. Find the force between the point charge and the sphere. (Hint: again, find an image charge outside the sphere with negative charge.)

3. (25 %) [Magnetostatics]

   A hollow sphere with magnetic permeability μ has inner radius *a* and outer radius *b*. The sphere is placed in a uniform magnetic field B_0. Find the field inside the sphere and conclude this can be used for magnetic field shielding when $\mu >> \mu_0$. (Hint: Since no current is present, we can define a scalar potential ψ_m such that $\vec{B} = \nabla \psi_m$, and $\nabla^2 \psi_m = 0$. Due to ϕ symmetry, the general solution is $\psi_m = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos\theta)$. Then apply continuity conditions for $\vec{B}$ and $\vec{H}$ at $r = a$ and $r = b$. In this case, only $l = 1$ exists.) In spherical coordinate: $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$, and the Legendre polynomials are $P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1)...etc$

4. (25 %) Explain the following items:

   (a) Lorentz gauge and Coulomb gauge.
   (b) Kramers-Kronig relations.
   (c) Linear and circular polarization of light.
   (d) Rayleigh scattering.
   (e) Multipole expansion of electric potential.
   (f) Maxwell stress tensor.
   (g) Electric polarizability and magnetic susceptibility.
   (h) Synchrotron radiation.