Qualification Exam for PhD Candidates (spring 2013)
Classical Mechanics

You should derive your results step by step.

1. (15) Consider the elastic scattering of two particles with masses M and m ($M > m$). The speed of M is V and the particle m is at rest. Find the maximum scattering angle θ of M after collision.

2. (15) A particle with rest mass M decays at rest into another particle with rest mass m and two photons. Calculate the maximum kinetic energy of the particle with mass m in the theory of relativity.

3. (15) A bucket of water is set spinning about its symmetry axis with constant angular velocity ω. Determine the shape of the water in the bucket.

4. (15) Write down the Hamiltonian for a single particle with potential energy U in three-dimensional Cartesian, cylindrical and spherical coordinates. No derivation is required.

5. (15) Consider a damped oscillator. An external driving force is applied to the oscillator. The equation of motion is

$$d^2x/dt^2 + 2\beta dx/dt + \omega_0^2 x = A\cos(\omega t).$$

Determine the particular solution of this equation.

6. (10) Consider a string with linear mass density ρ (mass per unit length) and tension force τ. (a) Write down the wave equation for $q(x,t)$, where q is the displacement of the string. (b) What is the general solution of this wave equation? No derivation is required.

7. (15) A sphere of radius r is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of radius R. Determine the Lagrangian function and the Lagrange's equation of motion. Find the frequency of small oscillation.

Qualifying examination – Statistical Mechanics

Spring, 2013

1. Briefly explain the physical meaning of following items (20 points)
   (a) Basic assumption in Statistical Mechanics
   (b) Canonical ensemble and Grand canonical ensemble
   (c) Bose-Einstein condensation
   (d) Fermi pressure
   (e) Phonons

2. (About partition function, Gibbs paradox, ideal classical gas, and entropy, 20 points)
   (a) Write down the partition function for an ideal classical gas of N identical particles of individual mass m, confined in a container of volume V and temperature T. (5 points)
   (b) Solve for the corresponding free energy and use it to derive the chemical potential μ, pressure P, and entropy S. (9 points, You may need the formula for the Gaussian integral, $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$)
   (c) Take two such containers of the same gas and open the partition to allow them to mix. What is the entropy change after the mixing? (6 points) Explain the results you have)

3. (About quantum statistics, black-body radiation, 20 points)
   (d) It is known that photons obey Bose-Einstein distribution $f_{\mathrm{BE}}(\varepsilon) = \dfrac{1}{e^{\beta\varepsilon} - 1}$ where $\beta \equiv 1/k_B T$ and chemical potential $\mu = 0$. $\varepsilon = pc$ is the photon energy of momentum p (c is speed of light). Write down the total energy stored in the electromagnetic waves inside a cubic black box of length L and temperature T.
   (e) Show that the total energy emitted in all frequencies per second per cm^2 from the surface of this black box is proportional to T^4 (you do not need to do the full calculation).

4. (About phonons, Debye and Einstein models, 20 points)
   (f) Describe Einstein's theory of the heat capacity of a solid, and explain why his theory disagrees with the experimental data for real crystals at very low temperatures. (10 points)

(g) Describe Debye's alternative construction of the same problem, and show his prediction for the heat capacity at very low and very high temperatures, respectively. (10 points)

5. (About Fermions, quantum pressure, 20 points)

(h) Show that the Fermi pressure for an ideal Fermi gas obeys $P = 2U/(3V)$ where U denotes the internal energy and V is the volume.

(i) Derive P as a function of the particle density n at zero temperature.

Quantum Mechanics Qualification
Feb. 24, 2013.

1. Answer the following questions briefly(5+5+5+5+5%)
   (a) What are the generators of spatial translations?
   (b) What are the Hermitian conjugates of operators $x^2\frac{d}{dx}$ and $e^{-i|\alpha\rangle\langle\beta|}$?
   (c) For a wavefunction $\psi = \frac{1}{\sqrt{3}}Y_2^{-1}(\theta,\phi) - \frac{\sqrt{2}}{\sqrt{3}}Y_3^{-1}(\theta,\phi)$, find $\langle\psi|L_+|\psi\rangle$ and $L_z\psi$.
   (d) A recent hot news was a bright meteor (shooting star) flying across the sky over Chelyabinsk, Russia. What is the physics and quantum processes behind the flare?
   (e) What is Aharonov-Bohm effect?

2. (10%) A system consists of two particles with spins $s_1 = 3/2$ and $s_2 = 1/2$. Find all possible eigenvalues of the operator $(\mathbf{s}_1 - \mathbf{s}_2)^2$.

3. (15%) Consider two particles of the same mass m in one dimension and they are connected by a spring with spring constant k. Suppose that the total momentum of the system is p, find all possible total energies for the following cases: (i) two particles are different (ii) two particles are identical fermions (iii) two particles are identical bosons.

4. (10%) Consider a one-dimensional harmonic oscillator with mass m and natural frequency ω. The oscillator is in its ground state $|0\rangle$ at $t = -\infty$. A time dependent potential is switched on as follows

$$V(x,t) = \frac{V_0 x}{1 + (t/\tau)^2}$$

   where τ is a positive constant and V_0 is a constant as well.
   Find the probability that the oscillator is in the state $|n\rangle (n \neq 0)$ at $t = \infty$ to order of V_0^2.

5. (10%) Consider a central scattering potential $V(r) = V_0$ for $r < a$ and $V(r) = 0$ elsewhere, where V_0 is a constant. Use the Born approximation to evaluate the total scattering cross section in the limit of ($ka \ll 1$), where k is the momentum of the incident particle.

6. (15%) A perfectly elastic steel ball of diameter 1 cm is dropped vertically from a height 10 cm above an identical ball which is anchored to the ground. Taking into account the uncertainty principle, estimate the maximum number of bounces that can occur. (The density of iron is around $\sim 8g/cm^3$.)

7. (15%) A point particle of mass m is subject to the following central potential

$$V(r) = \begin{cases} 0, & 3a < r < 5a \\ \infty, & \text{elsewhere}\,. \end{cases}$$

   Find the corresponding eigen-energy and the normalized wavefunction for $l = 0$. Recall that the Laplacian operator is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

   in the spherical coordinate.

Ph.D. Qualify Exam: **Classical Electrodynamics** February 24, 2013

1. (20%) Write down the appropriate equations and explain the physical meanings of the following terms except (e), for which please indicate the most useful situations:
(a) Lorentz force law for a point charge in presence of electric and magnetic fields.
(b) Maxwell's equations.
(c) Poynting's vector and Poyting's theorem.
(d) Group velocity and phase velocity.
(e) Coulomb gauge and Lorentz gauge.

2. (20%) (a) Describe in words and qualitatively draw the field lines pictures of the electric field and magnetic field of a point charge moving with constant velocity. (b) Make a qualitative drawing of the emitted radiation power profile from a accelerating point charge when its velocity is (i) zero; (ii) parallel to acceleration; (iii) perpendicular to acceleration.

3. (20%) The potentials of two thin concentric shells, of inner radius a and outer radius b, are given by $V_a(\theta,\phi)=0$ and $V_b(\theta,\phi)$ respectively. Assume that there are no charges elsewhere. Find the potential at $r < a$, $a < r < b$, and $r > b$ for (a) $V_b(\theta,\phi)=V_0$, where V_0 is a constant; (b) $V_b(\theta,\phi)=V_0\cos\theta$.

4. (20%) A long cylinder of radius R, made of linear isotropic material of magnetic susceptibility χ_m, is placed in a uniform external magnetic field $\vec{B}_0$ with the field direction along the axis of the cylinder. Find: (a) the magnetization of the cylinder; (b) the magnetic field inside the cylinder; (c) the magnetic field outside. (Note that you can use the approximation that the cylinder is infinitely long.)

5. (20%) A current flowing in a short and thin straight wire of length ℓ is described by $I(t)=I_0e^{j\omega t}$, where the physical current is the real part of $I(t)$. Let us assume the short wire lies along the z-axis with the center of the wire at the origin. Obtain (a) the vector potential $\vec{A}(\vec{r},t)$; (b) the $\vec{B}$ field; (c) the $\vec{E}$ field; and (d) a drawing qualitatively depicting the radiation profile. (Note that keep only the terms proportional to $1/r$.)