Qualify Exam for Classical Mechanics

1. (20%) Consider two coupling harmonic oscillators with the same magnitude of instrinsic spring constant k but with different masses $m_2 = 2\ m_1$. The potential energy is given by

$$V(x_1, x_2) = \frac{k}{2}(x_1^2 + x_2^2) + \epsilon x_1 x_2\ ,$$

where x_1 and x_2 are the coordinates of particle 1 and particle 2 respectively. Calculate the frequencies of the eigen modes. Hint: First make a transformation so that the kinetic energy terms are isotropic!

2. (20%) Calculate the minimum energy needed for an incident photon γ to hit an electron at rest, producing an electron-positron pair:

$$\gamma + e^- \rightarrow\ e^-\ +\ e^+\ +\ e^+\ .$$

The energy of the electron may be put at $0.5\mathrm{MeV}/c^2$.

3. (20%) The potential energy of an anharmonic oscillator with mass m is given by

$$V(x) = \frac{k}{2}x^2 + \epsilon x^4$$

where x is the coordinate of the oscillator. (1). What is the Hamilton's equation for the system? (2). Assuming that ϵ is small, calcuate the correction to the frequency for small oscillations up to lowest orders in ϵ.

4. (20%) A stone falls without initial velocity into a 300 m deep mine shaft at a latitude of 60° north. How far does it deviate from the vertical?

Qualifying examination – Statistical Mechanics

Fall, 2012

1. (main concepts in statistical mechanics, 30 points)
   (a) What are the fundamental assumptions of equilibrium statistical mechanics?
   (b) Distinguish the following energies: Helmholtz $(F \equiv U - TS)$ free energy, grand potential $(-PV)$, Gibbs $(G \equiv U - TS + PV)$ thermodynamic potential, and enthalpy $(H \equiv U + PV)$.
   (c) Gibbs' paradox and how one resolves the unphysical puzzle.
   (d) Bose-Einstein condensation.
   (e) Determine the critical mass of a neutron star (called the Chandrasekhar limit) beyond which the gravitational pull can overcome the quantum pressure and cause further shrinkage of the dead star into a white dwarf.

2. (entropy, 20 points)
   What is the entropy of the following three-dimensional systems consisting of N atoms with atomic weight m and at temperature T?
   (a) A crystal. At very low temperature, the interaction between these atoms can be neglected. Assume the ground state of each atom is doubly degenerate.
   (b) An ideal classical gas.
   (c) An ideal Fermi gas with T much less than the Fermi temperature.

3. (specific heat, 30 points)
   Calculate the specific heat of the following systems at low temperature.
   (a) A classical molecular gas consisting of identical atoms with atomic weight m. Assume that the gas is dilute so that intermolecular interactions can be neglected. To consider rotations, use I to denote the moment of inertia of each molecule. In the mean time, approximate the bonding between atoms by a spring constant K to describe the vibrating motion. (10 points)
   (b) A solid in the Einstein model; e.g., N number of independent oscillators with intrinsic harmonic frequency ω. (5 points)
   (c) A one-dimensional solid in the Debye model; e.g., N atoms with atomic weight m are connected in series by springs with spring constant K. For simplicity, adopt the periodic boundary condition; that is, arrange these atoms in a circle with one extra spring connecting the first and last atoms. (15 points)

4. (one-dimensional Ising model, 20 points)

The Hamiltonian of one-dimensional Ising model is

$$H = -J\sum_{i=1}^{N-1} S_i S_{i+1} - B\sum_{i=1}^{N} S_i$$

where $S_i = \pm 1$ and B is the external magnetic field. When the coupling constant J is positive/negative, it favors parallel/antiparallel alignments for the spins.

(a) Please show that the magnetization, $M = \sum_{i=1}^{N} \langle S_i \rangle$ where $\langle\ \rangle$ denotes the statistical average, is always zero in the absence of B. In other words, there is no phase transition into a magnetic state at any temperature T.

(b) Find the magnetic susceptibility, $\chi \equiv \dfrac{dM}{dB}$, when B is much less than both J and $k_B T$.

Quantum Mechanics

Useful information

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2+\beta x} = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/4\alpha}$$

Problem 1 20% Answer the following questions *briefly*

(a) 5% What are generators of rotation?

(b) 5% Find Hermitian conjugates of the operators: $x\frac{\partial}{\partial x}$ and $4^{|\alpha\rangle\langle\beta|}$.

(c) 5% Consider 6 identical Fermions in one dimension. Let x_i and p_i ($i = 1, 2, 3, ...6$) be the corresponding position and momentum operators for 6 particles. Which of the following operator(s) is(are) observable(s)?

$p_1 + p_2 + ...p_{10}$, $x_2^2 + x_4^2 + x_6^2$, $\sum_{i<j} \frac{e^2}{4\pi\epsilon_0(x_i - x_j)}$.

(d) 5% Let $\hat{\mathbf{P}}$ and $\hat{\mathbf{L}}$ be the three dimensional momentum and orbital angular momentum operators. Consider the operator

$$\exp\left(\frac{i\hat{\mathrm{J}}_x\phi}{\hbar}\right)\hat{\mathrm{P}}_y\hat{\mathrm{L}}_z\exp\left(-\frac{i\hat{\mathrm{J}}_x\phi}{\hbar}\right).$$

Here $\hat{\mathbf{J}}$ is the total angular momentum operator. Express the above operators in terms of $\hat{\mathrm{P}}_y$, $\hat{\mathrm{P}}_z$, $\hat{\mathrm{L}}_y$, $\hat{\mathrm{L}}_z$, and ϕ.

Problem 2 22% Consider a particle at one dimension. Let the mass of the particle be m and its position is described by x.

(a) 9% If at $t = 0$, the particle is found precisely at $x = 0$ and the particle is other free for $t > 0$, find the wavefunction $\psi(x, t)$ of the particle at a later time t.

(b) Suppose that at $t = 0$, the particle is not so precisely at $x = 0$ but is located at $x = 0$ with the wavefunction being given by $\psi(x, 0) = (\pi a^2)^{-1/4}\exp(-x^2/2a^2)$, where $a > 0$. The particle is otherwise free for $t > 0$.

(i) 6% Find the position operator in the Heisenberg picture $\hat{\mathrm{x}}_H(t)$ (for $t > 0$) in terms of the operators $\hat{\mathrm{x}}$ and $\hat{\mathrm{p}}$ defined in the Schrodinger's picture. Calculate the commutator $[\hat{\mathrm{x}}_H(t), \hat{\mathrm{x}}_H(t')]$.

(ii) 7% Using results from (i), calculate $\Delta x(t)$ for $t > 0$.

Problem 3 31% Suppose that the Hamiltonian of a particle is given by

$$H = 5a^\dagger a + \beta a^2 + \beta(a^\dagger)^2 + 3$$

where after appropriate choice of units ($\hbar = 1$ and etc), $[x, p] = i$ and a and $a^\dagger$ are given by

$$a = \frac{1}{\sqrt{2}}(x + ip) \text{ and } a^\dagger = \frac{1}{\sqrt{2}}(x - ip).$$

(a) 9% When $\beta = 2$, if one performs ideal measurements on the energy of this particle, what possible values he would get?

(b) 6% Find the normalized ground state wavefunction $\tilde{\phi}_0(x)$ for the case $\beta = 2$.

(c) 8% Suppose that in addition to H with $\beta = 2$, a perturbed potential $V = \alpha x^4$ is applied to the particle. Find the energy shifts for all energy levels to the order of α.

(d) 8% Suppose that β is switched on as follows: $\beta(t) = \frac{\beta_0}{1+(t/\tau)^2}$, where τ is a positive constant. At $t = -\infty$, the particle is in its ground state. (i) Find the probability that the particle makes transition to an excited state at $t = \infty$ to the order of β_0^2.

(ii) For $\beta_0 = 2$ and to $O(\beta_0)$ in the wavefunction, find the probability of finding the particle in the ground state of $\beta = 2$ (i.e., the ground state found in (b)) at $t = \infty$. Note that literally, the propability is greater than one due to trucation but do not worry about it.

Problem 4 (a) 4% Find $\mathbf{L} \cdot \mathbf{S} \left(Y_1^{-1}(\theta, \phi) + 2Y_1^0(\theta, \phi) \right) |+\rangle$ in terms of $Y_l^m(\theta, \phi)$, $|+\rangle$, and $|-\rangle$, where $\mathbf{L}$ are the orbital angular momentum operator and $\mathbf{S}$ are the spin operators.

(b) 4% A beam of unploarized spin-1/2 particles, moving along the y-axis, is incident on two collinear Stern-Gerlach apparatuses, the first with $\mathbf{B}$ along the z axis and the second with $\mathbf{B}$ along the z' axis, which lies in the x-z plane at an angle $\pi/3$ relative to the x axis. Both apparatuses transmit only the uppermost beams, what fraction of particles leaving the first will leave the last?

(c) 4% Consider a system of two particles with spins $s_1 = 3/2$ and $s_2 = 1/2$. By considering the addition of two spins, find all possible eigenvalues to square of the difference of spin operators $(\mathbf{S}_1 - \mathbf{S}_2)^2$.

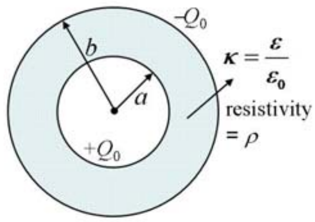
Problem 5 10% Consider the mutual elastic scattering of two spin-1/2 fermions. The Hamiltonian for this system is $\hat{\mathrm{H}} = \frac{\hat{\mathrm{p}}_1^2}{2m} + \frac{\hat{\mathrm{p}}_2^2}{2m} + \mathrm{V}(\mathbf{r}_1 - \mathbf{r}_2)$, where $\mathrm{V}(\mathbf{r}_1 - \mathbf{r}_2) = g\delta(\mathbf{r}_1 - \mathbf{r}_2)$. In the lab frame, the scattering is set up in the way that one fermion is initially at rest, while the other one is incident with a momentum $\hbar\mathbf{k}$. Both fermions are not polarized. Use the Born approximation to calculate the differential cross section $\sigma(\theta, \phi)$ in the center of mass (CM) frame. What is the differential cross section in the lab frame? What would be the differential cross section in the CM frame if these particles are identical fermions?

Ph.D. Qualify Exam

Classical Electrodynamics

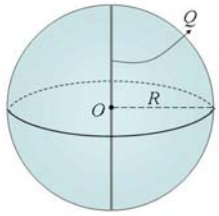
Fall 2012

1. (25 %) A hollow sphere made of dielectric material with dielectric constant κ and resistivity ρ has inner radius a and outer radius b as shown in the right figure. At time $t = 0$, positive charge $+Q_0$ and negative charge $-Q_0$ are placed uniformly on the inner and outer surfaces.



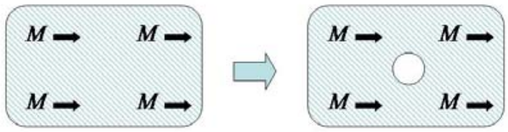
   (a) At $t = \infty$, find the charge distribution on the sphere.
   (b) Between $t = 0$ and $t = \infty$, how much Joule heat is dissipated due to the charge flow?
   (c) Show that the charge on the inner surface decays exponentially and determine the time constant.

2. (20 %) Consider a hollow grounded sphere of radius R in which there is a uniform line charge of total charge Q located on the z-axis between the north and south poles.



   (a) Verify the charge density within the sphere ($r < R$) can be written as: (in spherical polar coordinates)
   $\rho(r,\theta,\phi) = \frac{Q}{R}\frac{1}{4\pi r^2}\left[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right]$, where δ is the Dirac delta function with normalization condition: $\int_0^{\pi} \delta(\cos\theta \pm 1)\cdot d(\cos\theta) = 1$.
   (b) Suppose the potential inside the sphere has been calculated as:
   $\Phi(r,\theta,\phi) = \frac{Q}{4\pi\varepsilon_0 R}\left\{\ln(\frac{R}{r}) + \sum_{l=1}^{\infty}\frac{4l+1}{2l(2l+1)}\left[1 - \left(\frac{r}{R}\right)^{2l}\right]P_{2l}(\cos\theta)\right\}$, find the surface charge density on the sphere.

3. (25 %) A large magnet originally has a uniform magnetization M (magnetic moment per unit volume). (a) Find the magnetic field B inside the magnet. (Hint: you can start from a uniformly magnetized sphere and then make the radius very large.) (b) Now a small hole with radius R is cut inside the magnet. Find the magnetic field inside and outside the hole. You may take the center of the hole as origin.



4. (30 %, 5 % each) Explain the following items qualitatively and quantitatively.
   (a) Lorentz gauge and Coulomb gauge.
   (b) Kramers–Kronig relations.
   (c) Electric polarizability.
   (d) Circular polarization of light.
   (e) Diamagnetic and paramagnetic substances and explain the reasons causing their properties.
   (f) Explain why the sky is blue during the daytime and is red at sunset.