Qualification Exam [Classical Mechanics] (Feb. 2012)

1. (5%) A car moves on a straight road with a constant speed of $50km/h$. The radius of each tire is $50cm$. The tires roll without slipping. Calculate the acceleration of the point on the top of the tire.

2. (10%) What is the total cross section for the elastic scattering of a beam of particles of radius r from a fixed solid sphere with radius R?.

3. (15%) Consider two coupling oscillators with the same mass m and spring constant k. The potential energy is

$$V = k(x_1^2 + x_2^2)/2 + \epsilon x_1 x_2$$

   where ϵ is the coupling constant and x is the coordinate of the oscillator. Find the eigenfrequencies of vibration.

4. (15%) a pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k. Find Lagrange's equations of motion. (Do not solve these equations.)

5. (15%) A photon γ hits an electron at rest, producing an electron-positron pair:

$$\gamma + e^- \to e^+ + e^- + e^-.$$

   Calculate the minimum energy of the incident photon. The electron's rest mass is 0.5 MeV/c^2.

6. (15%) A rain drop falls through a cloud collecting mass as it falls. Assume that the drop is spherical and that the mass increased at a rate proportional to the cross sectional area of the drop. If the drop starts from rest where it is infinitely small, what is the acceleration?

7. (10%) A particle of mass m and initial speed v_0 is scattered by a fixed center of force. The perpendicular distance between the center and the line along the initial velocity is d (impact parameter). The force is repulsive and the magnitude of this force is mc/r^2 where c is a positive constant and r is the distance of the particle from the center. Calculate the minimum value of r.

8. (15%) The potential energy of a particle is $V = kx^2/2 + bx^4/4$ where k and b are constants. Find Hamilton's equations of motion. (Do not solve these equations).

Quantum Mechanics

Problem 1 25% Answer the following questions *briefly*

(a) What is the most important consequence for a system being translational invariance?

(b) Write down the Dirac equation and indicate what kind of particles it is intended to describe.

(c) Consider a two-level system characterized by the Hamiltonian

$$\mathbf{H} = \Delta \left[|1><1| - 2|2><2| + 2|1><2| + 2|2><1| \right],$$

where Δ is a real number with the dimension of energy. $|1>$ and $|2>$ are normalized and are also orthogonal to each other. If one performs ideal measurements on the energy of this system, what possible values he would get?

(d) Consider 10 identical bosons in one dimension. Let x_i and p_i ($i = 1, 2, 3, ...10$) be the corresponding position and momentum operators for 10 particles. Which of the following operator(s) is(are) observable(s)?

$|p_1 - p_{10}| + |p_2 - p_9| + |p_3 - p_8| + |p_4 - p_7| + |p_5 - p_6|$, $x_1 + x_2 + ...x_{10}$, $p_2^2 + p_4^2 + p_6^2 + p_8^2 + p_{10}^2$, $\sum_{i<j} \frac{e^2}{4\pi\epsilon_0(x_i - x_j)}$.

(e) Suppose that in the Aharonov-Bohm (AB) effect experiment, the particle sources are point particles with energy E and charge q. If the interference pattern is shifted by Δx in comparison to the pattern without the magnetic field, find the magnetic flux that is behind the double-slit (assume that the seperation between two slits is d and the distance from the slits to the screen is L).

Problem 2 Consider two particles of the same mass m in one dimension. Two particles are connected by a spring with spring constant k.

(a) 15% Suppose that the total momentum of two particles be p, find all possible total energies for the following cases: (i) two particles are non-identical (ii) two particles are identical fermions (iii) two particles are identical bosons.

(b) 8% Suppose that in addition to being connected by a spring, two particles carry charges of q and $-q$ and move along x axis. A uniform electric field E is applied along $+x$ direction. If two particles are in an energy eigenstate, find the minimum of root-mean-square relative distance (i.e., minimum of $\sqrt{\langle (x_1 - x_2)^2 \rangle}$) between two particles.

Problem 3 20% Consider the coupling of three spin-1/2 particles. Let $|\alpha\beta\gamma\rangle$ denote the state when the first particle in ithe state $|\alpha\rangle$, the 2nd in $|\beta\rangle$ and the 3rd in $|\gamma\rangle$, where α, β, and γ are either $+$ (spin up) or $-$ (spin down).

(a) Construct all states with total angular momentum $J = 3/2$.

(b) By setting coefficients associated with states starting with $+$ (i.e., $|+\beta\gamma\rangle$) being positive, construct all states with $J = 1/2$. (Hint: add two spin-1/2 partciles first, and then include the 3rd particle. You may find the following formula useful: $J_\pm |j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle$)

Problem 4 15% A particle of mass m is described by the wave function

$$\psi_E(r, \theta, \phi) = A \exp(-r/a_0),$$

where r, θ and ϕ are spherical coordinates and a_0 is a positive constant. Assuming that ψ_E is an energy eigenstate in a central potential $V(r)$. and $V(r) \to 0$ as $r \to \infty$, find the energy E of this state, $V(r)$, and all other possible energies for this particle.

Problem 5 Consider a one-dimensional harmonic oscillator with mass m and natural frequence ω. The oscillator is in its ground state $|0\rangle$ at $t = -\infty$. A time-dependent potential is switched on as follows

$$V(x, t) = \frac{V_0 x}{1 + (t/\tau)^2},$$

where τ is a positive constant and V_0 is also a constant.

(a) 10% Find the probability that the oscillator is in the state $|n\rangle$ $(n \neq 0)$ at $t = \infty$ to the order of V_0^2.

(b) 7% Suppose that for $t \geq 0$, $V(x,t) = V_0 x$. Find the wavefunction ψ at $t = 0$ to $O(V_0)$ in the adiabatic limit $\tau \to \infty$. Find a static Hamiltonian H such that to $O(V_0)$, ψ is an eigenstate to H.

Ph.D. Qualify Exam: **Classical Electrodynamics**

Feb. 2012

1. (30%) Explain the following terms qualitatively and quantitatively.

(a) Lorentz gauge and Coulomb gauge (5%)

(b) Poynting theorem (5%)

(c) Skin depth (5%)

(d) Group velocity and phase velocity (5%)

(e) Retarded Green function (5%)

(f) Synchrotron radiation (5%)

2. (10%, 10%) Green function

(a) What are Green's first identity and Green's theorem?

(b) For a point charge outside a grounded conducting sphere, find the Green function $G(\mathbf{x},\mathbf{x}')$ that satisfies Dirichlet boundary condition. [Hint: the method of images.]

3. (10%, 10%) Two concentric conducting shells of inner and outer radii a and b (b>a), respectively. The outer shell is connected to a given potential V, while the inner shell is grounded $V(a,\theta)=0$.

(a) If $V(b,\theta)=V_0$ (constant), find the potential at $r<a$, $a<r<b$, and $r>b$.

(b) If $V(b,\theta)=V_0\cos^2\theta$, find the potential everywhere between the shells ($a<r<b$). [Hint: use Legendre polynomials $P_0(x)=1,\ P_1(x)=x,$ and $P_2(x)=(3x^2-1)/2$].

4. Find the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ generated by a charged particle q, if

(a) the charged particle is moving with a constant velocity $\mathbf{v}=v_0\hat{\mathbf{x}}$. (10%)

(b) the charged particle is moving with a constant acceleration $\mathbf{a}=a_0\hat{\mathbf{x}}$. (10%)

[Hint: Liénard-Wiechert potentials and fields for a point charge.]

5. If $\mathbf{E}$ and $\mathbf{B}$ are perpendicular in the laboratory and $|E|=2|B|$, find a reference frame such that the field is pure electric or magnetic? If yes, what is the velocity of the reference frame relative to the laboratory? (10%)

[Hint: Gaussian unit $\mathbf{E}'_{\|}=\mathbf{E}_{\|},\ \mathbf{E}'_{\perp}=\gamma_0\left(\mathbf{E}_{\perp}+\frac{\mathbf{v}_0}{c}\times\mathbf{B}_{\perp}\right)$ and $\mathbf{B}'_{\|}=\mathbf{B}_{\|},\ \mathbf{B}'_{\perp}=\gamma_0\left(\mathbf{B}_{\perp}-\frac{\mathbf{v}_0}{c}\times\mathbf{E}_{\perp}\right)$]

Statistical Mechanics

Qualifying examination

Spring, 2012

1. （一般性的統計問題，15 分）
   (a) 隨意在長度爲 L 的繩子上砍 N 刀，請問當 N 遠大於 1 時，在距離繩子左端 ℓ 到 $\ell + d\ell$ 距離的小繩段內，剛好被切一刀的機率爲何？
   (b) 延續(a)小題，如果我指定這一刀是從左邊數起的第 n 刀，那(a)問的機率會變成什麼？
   (c) 把這 N+1 條繩段的長度 ℓ 和個數畫成柱狀圖表(histogram)後，fitting 出來的分佈機率函數 $P(\ell)$ 爲何？

2. (應用在生物物理的統計觀念，September 2010 考古題，20 分)
   (a) 如果忽略高分子(polymers)的每一小段 monomers 之間的作用力，爲了讓熵(entropy, S)達到最大，polymer 會捲曲成像顆蓬鬆的棉花球；請證明在溫度 T 的環境裡，包含 N 段（每段的長度爲 ℓ ）monomers 的 polymer 直徑會和 $\sqrt{N}\,\ell$ 成正比。當三維的 polymer 被壓扁成二維（或自行吸附在固體表面）時，這個直徑會變大幾倍？（提示：可以把這個題目轉化成醉漢走路(random walk)，N 變成步數， ℓ 則是步伐）
   (b) 當小孩長得像鄰居叔叔時，丈夫肯定會偷偷帶小孩去作親子鑑定，最常使用的方法是電泳(electrophoresis)實驗，檢驗師先用酶素把小孩唾液的 DNA 切成長短不一的線段，然後利用電場誘使它們鑽進（孔洞直徑大於 DNA 截徑，但是仍然小到限制它無法捲曲）孔洞結構的膠體裡，一旦 DNA 的端點露出洞的另一個開口，不再需要電場，爲了增大熵 S（相較於還留在洞裡的線段只有一種組態(configuration)，露出洞口的 DNA 線段則海闊天空可以隨意捲曲），DNA 自己會像蚯蚓般地慢慢鑽出來。從自由能的觀念來解釋，由於忽略 monomers 之間的位能，Helmholtz free energy 直接就等於 $-T\cdot S$ ，其中 T 代表環境溫度；當 S 增加，自由能便得以降低。用 N 和 ℓ 分別代表 monomers 數目和每段 monomer 的長度，請估計同樣從只露出一段 monomer 開始，不同長短的 DNA 鑽出孔洞需要費的時間如何隨 N 改變。

3. (古典統計力學的 Gibbs paradox，20 分)
   (a) 把外面裹有鋁箔紙的密封箱子隔成等體積的兩半，分別灌入相同顆數 N 和溫 T 的不同惰性氣體（原子量分別用 m_1, m_2 表示），當我把隔板上的小孔打開，讓兩邊的氣體得以緩慢、可逆地達到平衡，請問最終箱子的總

熵(entropy)會增加多少？

(b) 如果當初兩邊灌的根本是同一種氣體，那打開小孔的動作應該不會改變系統的狀態，也就是說熵必須保持定值，請問這個事實如何用你在(a)的推導方式得到？

4. （用到平均場近似的自旋問題，September 2009 維甫的考古題，25 分）

Consider a single electron in a magnetic field B along the z-axis. The Hamiltonian operator is $\hat{H} = -\mu B \hat{\sigma}_z$, where μ is the magnetic moment and the $\hat{\sigma}_z$ operator is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3}$$

(a) What is the density matrix of the system in the canonical ensemble?

(b) Derive the partition function of this system (Feb. 2006)

(c) Derive the average (per particle) magnetic moment, M, and calculate its asymptotic behavior in low and high temperature limit. (Feb. 2005)

(d) Now, assuming the magnetic field B is not given by a source outside the spin system, but is generated by the local spin moment itself, i.e. let $B = \alpha M$, where α is some constant coefficient. Then, derive the self-consistent equation of M, and calculate the critical temperature, T_c, of ferromagnetic transition, below which the self-consistent solution of M, defined by M_{MF}, is nonzero. This is so called mean-field solution.

(e) Assuming T_c is very high and one can use the high temperature expansion to the leading order correction, calculate M_{MF} as a function of temperature δT within the ferromagnetic phase, where $\delta T \equiv T_c - T$ is a small positive quantity. (hint: $\tanh(x) \sim x - x^3/3$ for small x)

5. （Equipartition 定理，September 2010 林克瀛教授的考古題，15 分）

Consider a classical ideal gas at temperature T. The energy of a molecule is

$$\epsilon = ax^2 + bx + \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) \tag{1}$$

where a and b are constants. Determine the average energy of a molecule.

6. （觀念題，Fall 2007 秀豪的考古題，10 分）

(a) Ergodic hypothesis

(b) Grand canonical ensemble

(c) Fermi pressure

(d) Liouville’s theorem