Qualification examination (fall, 2011)

- Classical Mechanics -

1. (Calculus of variations, 20%)
When minimizing a function, we differentiate it with respect to its free parameter. But if it is an integral that we want to minimize, the variable becomes a function and the equivalent action of differentiation is the variation. For classical actions $S \equiv \int_{t_i}^{t_f} L(q,\dot{q})dt$, the starting and end points $q(t_i), q(t_f)$ are fixed, and so only variation and no differentiation is needed. Let me now test you on a problem that requires both actions; namely, determine the shape and contact angle of a liquid drop on a solid surface. Use $\varepsilon_{1,2}, \varepsilon_{2,3}, \varepsilon_{1,3}$ to denote the amount of energies that can be lowered for each unit area between air/liquid, liquid/desk, and air/desk. Make sure you include the surface tension T and the constraint that the liquid volume needs to be conserved.
(a) Variate the drop shape over the total potential energy and write down the Euler-Lagrange equation of motion that determines the drop shape. No need to solve this differential equation.
(b) Differentiate with respect to the drop height and obtain a simple relation for the contact angle, which is often ascribed to the balance of forces.

2. (Hamiltonian equations of motion, Goldstein's Exercise 8.14, 20 points)
The Lagrangian for a system can be written as
$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y}^2 - k\sqrt{x^2+y^2} \ ,$$
where $a, b, c, f, g,$ and k are constants. What is the Hamiltonian? What quantities are conserved?

3. (Canonical transformation, Goldstein's Exercise 9.24, 20 points)
(a) Show that the transformation $Q = p + iaq,\ P = \dfrac{p - iaq}{2ia}$ is canonical and find a generating function.
(b) Use the transformation to solve the linear harmonic oscillator problem.

4. (2009 考古題, 加點變形, 20%)
A uniform ladder leans against a smooth vertical wall. The initial angle between the floor and ladder is θ_0. If the floor is also smooth, the ladder will slide down.

(a) Find the angle between the floor and the ladder when the ladder loses contact with the vertical wall.

(b) Find the shortest distance between the top of the ladder and the foot of the wall when the ladder lies totally flat on the floor.

5. (2008 考古題, 加點變形, 20 points)

A point particle of mass m and initial speed v_0 is scattered by another originally static point particle of mass M. The perpendicular distance between the center of force and the incident velocity is d (called the impact parameter). Assuming that the force is repulsive and the magnitude of this force is $\frac{mc}{r^2}$ where c is a positive constant and r is the distance between these two particles.

(a) Calculate the minimum value of r.

(b) Calculate the differential and total cross sections.

6. (2002 考古題，加分題，10 分)

A raindrop falls through a cloud collecting mass as it falls. Assume that the drop remains spherical and that the mass of the raindrop increases at a rate which is proportional to the cross-sectional area of the drop multiplied b the speed of fall. If the drop starts from rest when it is infinitely small, what is the acceleration?

Qualification Exam for Statistical Mechanics
(September 2011)

You should derive your results step by step clearly.

1. (10%) Consider a model of gas where all particles move with the same speed v in random directions. Calculate the average value of the relative speed.

2. (15%) An ideal classical gas composed of N particles, each of mass m, is enclosed in a vertical cylinder of height L, placed in a uniform gravitational field of acceleration g and is in thermal equilibrium. Evaluate the partition function of the gas.

3. (10 %) Estimate how long (in seconds) it takes a molecule of air to move to a position 5 meters away. Assume that the mean free path is 5×10^{-6} m and the average speed is 500 m/s.

4. (15 %) The atmosphere is often in a state of constant entropy (PV^γ =constant). In a uniform gravitational field, find (dT/dz) where T is the temperature at altitude z.

5. (10 %) Calculate the density of (quantum mechanical) states of a particle with mass m confined in a cubic box with volume V,

6. (15%) A classical harmonic oscillator of mass m and spring constant k is known to have a total energy E, but its starting time is not known. Find the probability density function, P(x), where P(x)dx is the probability that the mass would be found in the interval dx at x.

7. (10 %) A crystal consists of N identical atoms. At very low temperature, the interaction between atoms can be neglected. If the ground state of each atom is doubly degenerate, what is the entropy of the crystal at zero absolute temperature?

8. (15%) Consider the Ising model on a one dimensional open chain. The Hamiltonian is

$$H = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}$$

where $\sigma_i = 1$ or -1. Calculate the partition function

$$Q_N = \sum_{\sigma_1} \cdots \sum_{\sigma_N} \exp(-\frac{H}{kT}).$$

Quantum Mechanics Qualification, Sep. 18, 2011.

You must provide the details or reasonings to justify your answers.

1. (8+7%) The Hamiltonian of a system is given by

$$H = \frac{41}{9}a^\dagger a + \frac{20}{9}\left(a^2 + (a^\dagger)^2\right) + \frac{25}{9}$$

where

$$a = \frac{1}{\sqrt{2}}(q + ip)\,, a^\dagger = \frac{1}{\sqrt{2}}(q - ip)\,, \text{ and } [q, p] = i.$$

(1) What are the eigen-energies of this system? (2) What is the ground state wave function? Remember to fix the normalization.

2. (5+5%) A particle whose wave function is given by

$$\psi(\vec{r}) = \left(\frac{\sqrt{2}}{\sqrt{3}}Y_{11}(\theta,\phi) - \frac{\sqrt{11}}{\sqrt{6}}Y_{1,-1}(\theta,\phi) + \frac{1}{\sqrt{2}}Y_{10}(\theta,\phi)\right) f(r)$$

where $f(r)$ is a normalized radial function, $\int_0^\infty r^2 f^2(r)dr = 1$.
(a) What are the expectation values of $\hat{L}^2$ and $\hat{L}_z$?
(b) What is the expectation value of an operator $\hat{V}(\theta) = 3\cos^2\theta$ in this case?

You may find the following information useful:

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta\,,\; Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

3. (8+8%) (a) Write down the spatial and spinal wave function of the first excited state for two noninteracting electrons in an infinite potential well, $V(x) = 0$ for $0 \le x \le L$ and $V(x) = \infty$ elsewhere.
(b) Similarly, write down the spatial wave function of the first excited state for two noninteracting spin-0 particles in the same potential well.

4. (10%) An astronaut in his spacesuit (total mass of 200kg) is stranded 3m from the spaceship. He hands and shoot a 10 Megawatt laser directly away from the ship and leaves it shinning. About how long will it take him to move 3m and reach safety? He started from rest relative to the ship.

5. (5+5+10+5+5%) An electron is trapped in a 2-dimensional infinite potential well in a rectangular area $a \le x \le a + L$ and $b \le y \le b + 2L$. (a) Write down the corresponding Schrodinger wave function with proper normalization. (b) What is the probability of finding the ground state electron in the rectangular region

$a \leq x \leq a + L/3$ and $b \leq y \leq b + L$?
(c) If a perturbation

$$\Delta V(x, y, t) = \lambda \cos \frac{\pi(y - b)}{2L} \cos \frac{3\hbar\pi^2 t}{8mL^2}$$

is applied to the system, discuss the rate of transition from the ground state to the excited states.

If 10 more electrons are filled in and we assume there is no interaction among the electrons,
(d) what is the ground state energy of the 11-electron system? (e) What's the minimal photon energy to excite the system?

6. (9%) The Clebsch-Gordan coefficients are coefficients those are used for adding two angular momenta (j_1 and j_2) to get the final angular momentum (j), and be represented as $\langle j_1, m_1; j_2, m_2 | j\, m\rangle$. It is known that $\langle 3/2, 3/2; 1/2, -1/2 | 2\, 1\rangle = 1/2$. Find $\langle 3/2, -1/2; 1/2, -1/2 | 2\, -1\rangle$, $\langle 3/2, 1/2; 1/2, 1/2 | 2\, 1\rangle$, and $\langle 3/2, 3/2; 1/2, -1/2 | 1\, 1\rangle$.

7. (10%) Calculate the Born approximation to the differential and total cross sections for scattering a particle of mass m off the δ-function potential $V(\vec{r}) = g\delta^3(\vec{r})$.

Ph.D. Qualify Exam: **Classical Electrodynamics**

September 2011

1. (10%, 10%) Green function for the wave equation

(a) Find the Green's function $G(\vec{r},\vec{r}')$ which is the solution of $(\nabla^2+k^2)G(\vec{r},\vec{r}')=-\delta(\vec{r}-\vec{r}')$. [Hint: Assume no boundary surface and depend only on $R=|\vec{r}-\vec{r}'|$.]

(b) Use the results of (a) to obtain the retarded potential

$V(\vec{r},t)=\dfrac{1}{4\pi\varepsilon_0}\displaystyle\int\dfrac{\rho(\vec{r},t-|\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|}d^3r'$ from the wave equation $V(\vec{r},t)$ under Lorentz gauge:

$\nabla^2V(\vec{r},t)-\dfrac{1}{c^2}\dfrac{\partial^2V(\vec{r},t)}{\partial t^2}=-\dfrac{\rho(\vec{r},t)}{\varepsilon_0}$. [Hint: Express $V(\vec{r},t)$ and $\rho(\vec{r},t)$ as Fourier integrals.

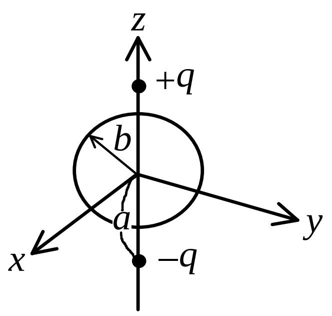
[Note: All quantities in this question are expressed in SI units]

2. (10%, 10%) Electrostatics

Two point charges $+q$ and $-q$ are located on the z axis at $z=+a$ and $z=-a$, respectively.

(a) Find the electrostatic potential as an expansion in spherical harmonics and powers of r for $r>a$ in the absence of the grounded sphere.

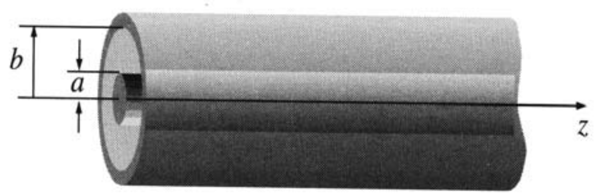
(b) If there is a grounded conducting sphere of radius b ($b<a$) centered at the origin, use the linear superposition to satisfy the boundary conditions and find the potential everywhere for $r \gg a$.



3. (10%, 10%) For a dielectric-filled $(\varepsilon=4\varepsilon_0,\ \mu=\mu_0)$ coaxial waveguide with inner radius a and outer radius b as shown below,

(a) find the electric and magnetic fields,

(b) and find the time-averaged energy density u and the energy flow **S** (energy per unit area per unit time) along the line.



4. (10%, 10%) Magnetostatics

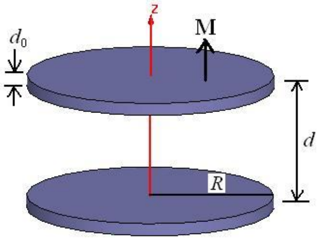
Consider two thin ferrite disks of radius R and thickness d_0, separated by a distance d ($>>d_0$). Assume the ferrite disks carry a uniform magnetization, $\mathbf{M} = M_r \hat{z}$.

(a) Find the bound surface currents $\mathbf{K}_b$ on the outer surfaces of the ferrite disk and the bound volume current density $\mathbf{J}_b$.

(b) Find the magnetic field at the midpoint of the central axis when d=R.

[Hint: The arrangement is similar to a Helmholtz coil.]

[Hint: $\mathbf{A}_M(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\nabla' \times \mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' + \frac{\mu_0}{4\pi}\oint_S \frac{\mathbf{M}(\mathbf{x}') \times \hat{\mathbf{n}}'}{|\mathbf{x}-\mathbf{x}'|} da'$].



5. (10%, 10%) Wave: skin depth

(a)Starting from the Maxwell equations, derive the dispersion relation (i.e. the relation between the wave frequency ω and the propagation constant k) for a plane electromagnetic wave in an infinite and uniform medium of conductivity σ, electrical permittivity ε, and magnetic permeability μ.

(b)Assume that the medium is a good conductor, derive an expression for its skin depth δ.

[vector formula: $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$]