NTHU Physics Qualifying Exam
Classical Mechanics 2026 Spring

Answer all questions. You must show your work. No credits will be given if you don't show how you get your answers.

1. (15 points) Consider motion in one dimension with the Hamiltonian

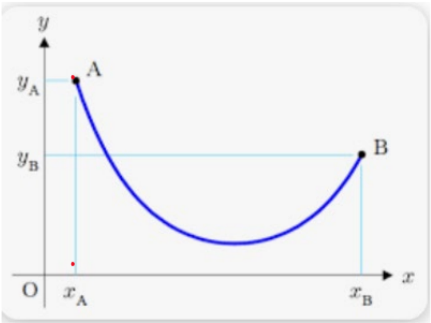
$$H = \frac{p^2}{2} - \frac{1}{2q^2}. \tag{1}$$

Show that

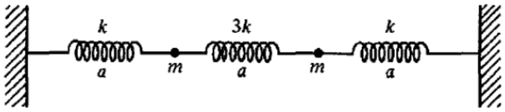
$$D = \frac{pq}{2} - Ht \tag{2}$$

is a constant of motion.

2. (15 points) A rope of indefinite length passes freely plane surface over pulleys of earth, at heights with a y_A and y_B above the plane surface of earth, with a horizontal distance $x_B - x_A$ between them. Find the curve between the pulley. Assume the rope has a uniform linear mass density.



3. (20 points) Two particles of mass m moves in one dimension at the junction of three springs, as shown in the figure. The springs all have unstretched lengths equal to a, and the force constants are $k, 3k, k$, respectively. Find the eigenfrequencies and normal modes of the system.



4. (20 points) Consider a photon of energy E_γ incident on a stationary proton. When enough energy is supplied by the photon, a π^0 meson can be produced in a reaction

$$\gamma + p \to p + \pi^0. \tag{3}$$

What is the minimum energy of the photon for this reaction to occur?

(please turn over)

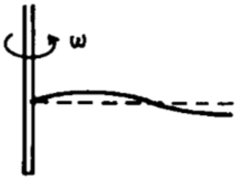
5. (30 points) Consider a piece of string of uniform mass density ρ and of length l. Denote the tension on the string by T.

   (a) Derive the equation for small oscillations of the string

$$\rho \frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left(T \frac{\partial y}{\partial x}\right) = 0 \tag{4}$$

   (b) Suppose the string is free at one end and is attached to a vertical pole at the other end, which is rotating about the pole at an angular frequency ω. Neglect gravity. Show that the equation for small oscillations reads

$$\frac{\partial}{\partial x}\left((l^2 - x^2)\frac{\partial y}{\partial x}\right) = \frac{2}{\omega^2}\frac{\partial^2 y}{\partial t^2} \tag{5}$$



   (c) Letting $\xi = x/l$ and doing a separation of variables $y(t,x) = e^{i\Omega t} f(\xi)$, show that the differential equation can be put in the form of Legendre

$$(1 - \xi^2)\frac{d^2 f}{d\xi^2} - 2\xi \frac{df}{d\xi} + \lambda^2 f = 0. \tag{6}$$

   Derive the eigenfrequencies of the small oscillations.

   Hint: It is known that the Legendre's differential equation has finite solutions known as Legendre polynomials only if λ^2 *takes quantized values of* $\lambda^2 = n(n+1)$ *for* n *being non-negative integers. The corresponding Legendre polynomial* $f = P_n(\xi)$ *has the property that* $P_{2n}(0) = \frac{(-1)^n}{4^n}\binom{2n}{n}$, $P_{2n+1}(0) = 0$.

Qualifying Examination – Statistical Mechanics

March 7-8, 2026

Please explain the logic behind your answers.

Problem 1. *Free fermion problem(10 points):* Let the density of states of the electrons in a system be assumed to be a constant D for $\varepsilon > 0$ ($D = 0$ for $\varepsilon < 0$). Here, ε represents the energy. We assume the total number of electrons be equal to N. Here $N = D\int_0^\infty \frac{1}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon$.

1. Calculate the chemical potential μ at $T = 0K$.
2. When the system is in the regime of quantum degeneracy, estimate or derive the expression of the specific heat and show that it is proportional to T.

Problem 2. *A simplified model of hemoglobin(10 points):* Hemoglobin is a protein in the blood that carries oxygen from the lungs to the muscles. It is formed by four units. Each unit can carry an O_2 molecule or not. For a hemoglobin, there will be 2^4 configurations as shown in Fig.1. (Each unit has two choices, carries nothing or an O_2 molecule. Therefore we have 2^4 possible choices.) As described in the caption of Fig.1, we can model the hemoglobin by assigning the binding energy $-\varepsilon_0$ and the additional gain, $-J$, if two nearby unit both bind with O_2(We try to pack O_2 to our hemoglobin as much as possible, a simple minded physicist guess). For example, considering the configuration in Fig. 2, the total energy of the simplified model is $(-\varepsilon_0) \times 2 + (-J) \times 1$. Consider we have $N/4$ hemoglobin, labeled by $\alpha = 1 \sim \frac{N}{4}$. On each hemoglobin, we have 4 units which can bind with O_2 molecules. We can use $\tau_{\alpha,i} = 0, 1$ for $i = 1 \sim 4$ to denote whether the i-th unit on the α-th hemoglobin is occupied by an O_2 molecule or not.

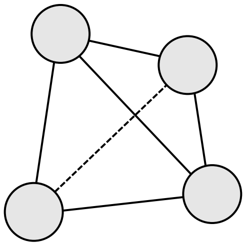


Figure 1: Schematic model of hemoglobin. The gray circles represent the unit that can carry an O_2 molecule. We assume the four units form a tetrahedral network structure. We can approximate the hemoglobin using a simplified model: The energy gain for an O_2 to bind with a unit is $-\varepsilon_0$. If two nearby units both contain O_2, the energy can further reduced by energy $-J$.

1. Write down the expression of the grand canonical partition function of the system with $\frac{N}{4}$ hemoglobin. Express the result using $x = exp\left[\frac{\varepsilon_0+\mu}{k_BT}\right]$and $y = exp\left[\frac{J}{k_BT}\right]$. (Hint: we

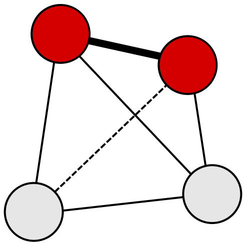


Figure 2: One particular configuration of hemoglobin. The red circles represent the units that are occupied by O_2 molecules. The thick line represent the additional energy reduction, $-J$, for nearby units are occupied by O_2 molecules. There will be other 5 configurations with equal energy.

consider the molecules of hemoglobin to be independent. So, we can factorize the partition function of the system into products of partition functions of individual hemoglobin molecule. Then, within each Hemoglobin, we can enumerate all the possible configurations and the corresponding energy to construct our partition function. Be careful about the number of configurations with equal energy. The degeneracy of energy in this system is not very regular, so one needs to specify them explicitly.)

2. Evaluate the average number $\langle M\rangle = \sum_{\alpha=1}^{N/4}\sum_{i=1}^{4}\langle\tau_{\alpha,i}\rangle$ of adsorbed molecules as a function of x, y.

Problem 3. *Bosons in Harmonic traps(30 points):* Let us consider a particle in the anisotropic harmonic-oscillator potential $V(\boldsymbol{r}) = \frac{1}{2}\left(K_x x^2 + K_y y^2 + K_z z^2\right)$. Therefore, we can consider the system as three independent harmonic oscillators in three different directions x, y and z. The corresponding energy levels can be parameterized by three non negative integers (n_x, n_y, n_z) as

$$E(n_x, n_y, n_z) = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z. \tag{1}$$

1. Let us consider n_i are continuous variables and neglect the zero-point energies (The $\frac{1}{2}\hbar\omega_i$ in $E(n_x, n_y, n_z)$.), we can simplify the energy as $E \approx \varepsilon_x + \varepsilon_y + \varepsilon_z$ where $\varepsilon_i = n_i\hbar\omega_i$. If we define $G(E)$ as the number of states below energy a specific energy E. Derive the expression of $G(E)$ in terms of E, $\omega_x, \omega_y, \omega_z$ and $\hbar$. (Hint: There are several way to derive this result. One approach is to consider the problem in the (n_x, n_y, n_z) space. What is the geometric meaning of the states below energy E?)

2. Another way to understand density of state is $g(E) = \frac{dG(E)}{dE}$. Use the above expression to get the density of states.

3. We can evaluate the particles in the excited states using

$$N_{ex} = \int_0^\infty dE g(E)\frac{1}{e^{E/k_BT} - 1}. \tag{2}$$

Here, we assume N_{ex} reaches its maximum for $\mu = 0$. The definition of the transition temperature, T_c, of Bose-Einstein condensation is when the total number of particles can be just accommodated in excited states. That is,

$$N = N_{ex}(T = T_c, \mu = 0) = \int_0^\infty dE g(E)\frac{1}{e^{E/k_BT} - 1}. \tag{3}$$

$\int_0^\infty dx \frac{x^{\alpha-1}}{e^x-1} = \Gamma(\alpha)\zeta(\alpha)$. Here, $\zeta(\alpha) = \sum_{n=1}^{\infty} n^{-\alpha}$ is the Riemann zeta function and $\Gamma(\alpha)$ is the gamma function. Derive the transition temperature of this system and express the result using $N, \omega_x, \omega_y, \omega_z$ and the Riemann zeta function.

Problem 4. *Freely Jointed Chain (Gaussian model) and its mean field picture (40 points)*

Polymers are large(high molar mass) molecules composed of a large number of monomers bonded together to form a chain. In reality, the monomers are bounded covalently. That is, the potential energy between two monomers should be a complex function of the bonding angle between two nearby monomers. Let's make some assumptions to simplify our problem which leads to the freely jointed model. (We might over simplify the problem but let's just try it first to get a feeling about what kind of problem we are facing. Then we can ask how to put back more ingredients to make our analysis more realistic.)

- We assume the joint of the monomers can rotate freely, *i.e.* the potential energy is independent of the bonding angle, ϕ, between two nearby monomers.
- We assume the monomers can overlap with each other in space and have no interaction between each other.

The model with above two assumptions is the freely jointed chain model. It is the idealization of polymers analogous to ideal gas model for gases.

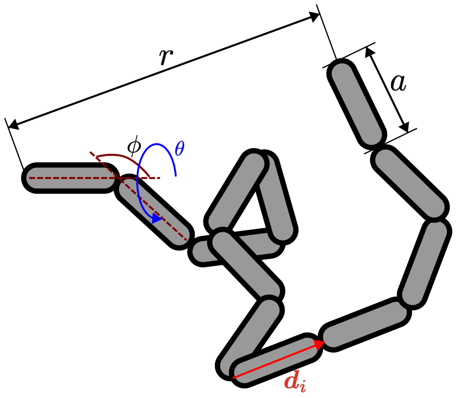


Figure 3: Schematic picture of a single chain polymer molecular formed by the monomers represented by the grey oval.

A key property that we are interested in is the size of the polymer. Usually the polymer coils up. The way they coils up will be a competition between the entropy and the energetics. Therefore, the size of the polymer is not simply the number of monomers, N, times the size of the monomer, a. Instead, we need to use some statistical description to characterize the size of the polymer. Usually, we use the root-mean-square end-to-end distance, $\langle r^2 \rangle^{1/2}$, as a measure for the size of the polymer. Here, r is the distance from one end of the polymer to the other in three-dimensional space. The average is taken over all possible ways the polymer coils. One of the configuration is simple, if all the monomers are aligned in one direction, the end-to-end distance $r_{straight} = Na$. However, this is just one of the possible value of r, once the polymer coils, $\langle r^2 \rangle^{1/2} < Na$.

1. It is interesting to observe that the freely jointed model in one-dimension is actually the simple one-dimensional random walk we discussed during the lecture. The step size is just the size of the monomer. The one-dimensional constraint restricts $\phi = 0/\pi$. Let's start in this

simple limit and use x to represent the end-to-end distance in one-dimensional case. Derive the probability distribution $P_{1D}(x,N)$ when the number of monomers N is large.

2. For the three-dimensional case, we assume the orientation is completely random and the vector from monomer i and the vector for monomer $i+1$ are uncorrelated. *i.e.* $\langle \boldsymbol{d}_i \cdot \boldsymbol{d}_j \rangle_{i \neq j} = 0$. Therefore, we expect N to be distributed evenly $N_x = N_y = N_z = \frac{N}{3}$. Here, N_x is a rough definition of the monomer belongs to the monomer in x direction. We can simply consider the projection of the monomer to the x, y, z direction. If the projection to x direction is has the largest size, we said it is a x monomer that should be counted in N_x. Derive $\langle r^2 \rangle^{\frac{1}{2}}$ and express it using N and a. This is a simple estimation for the size of the polymer. (Sometimes you will see people use the radius of gyration r_g^2 to estimate the size of the polymer. r_g is the average distance between monomers and the center of mass. It turns out the length scale, $\langle r_g^2 \rangle$, will be proportional with $\langle r^2 \rangle$. We will not discuss the calculation here, but just mention the fact that using the root mean square of the end-to-end distance capture the essential information for the size of the polymer.)

3. Now we have a rough idea about the freely jointed model. Let's assume we are in three-dimensional world. In the last problem, we should be able to have the probability distribution function $P_{3D}(r,N)$. Using the microcanonical ensemble, we should expect the probability distribution to related to the phase space volume as

$$P_{3D}(r,N) = \frac{\Omega(r,N)}{\int dr \Omega(r,N)} \equiv \frac{\Omega(r,N)}{A_N}. \tag{4}$$

Here, $\Omega(r,N)$ is the phase space volume for the size N polymer with the end-to-end distance r. A_N is the integral of $\Omega(r,N)$ over r. Use (4) to find the expression of the entropy, $S(r,N)$, for the size N polymer with the end-to-end distance r using N, a, r, k_B and A_N.

4. With the expression of entropy, we can derive the Helmholtz free energy of the system using $F(r,N) = U(r,N) - TS(r,N)$. Here, we are considering the freely jointed model, so there is no internal energy. The Helmholtz free energy is just $F(r,N) = -TS(r,N)$. Here is the interesting part, the free energy depends on r just like the spring with elastic energy $(E \propto k(\delta x)^2)$. We can have the elasticity constant of the polymer. Find the expression of the elasticity constant k for the polymer described by the freely jointed model.

5. A rubber is an composite object with entangled polymers. They are difficult to anlayze during the qualify exam. However, from the simple calculation we achieved here we can explain some interesting physics. Consider a piece of rubber hanging a massive object below it. Suppose the rubber is strong enough such that the massive object just hang in the air. Next, if we heat up the rubber (hopefully we did not melt it.), what would you expect? Will the massive object be lifted higher? Or the other way around?

6. Till now, we assume the non-interacting limit. So the free energy only consider the entropic effect. Now, we would like to extend the understanding to the case that the monomers can actually interact with each other. In order to proceed, we make the following assumptions:
   - In Fig. 3, we can clearly see the monomer is a rod like object which does not have rotational symmetry (it is not a sphere). However, **let's put that aside and assume the monomers are spherically symmetric objects with volume v that forbidden the overlap of two monomers.**

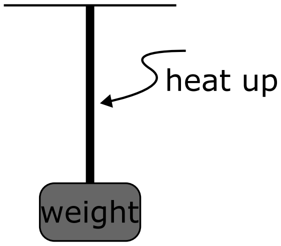


Figure 4: Schematic plot for the massive object hanging in the air.

- We can define the packing ratio, μ, of the monomers for the polymer as

  $$\mu = \frac{Nv}{r^3}. \tag{5}$$

  We drop the irrelevant coefficients just to get an estimation about the physics. It is the ratio of the total volume of the monomers and the volume given by the "size" of the polymer.

Since the interaction avoids two monomers to overlap in space. We can assume that the interaction between the monomers are pair-wise interaction. (They are just one monomer interacts with another one to satisfy the no-overlapping condition.) Estimate the interaction energy, $U_{int}(r, N, T)$, at temperature T for a polymer with length N at large N. (Hint: Think about the meaning of the packing ratio. How to estimate the pair-wise interactions using the packing ratio?)

7. Before we do any calculation, let's think about the effect of each term. The entropic contribution favors the value of small r to minimize the improved free energy. On the other hand, since the monomers repel each other due to the interaction energy, the $U_{int}(r, N, T)$ term prefers the value of large r in order to minimize the $F_{improved}(r, N, T)$. Therefore, the two effects competes with each other and fix the characteristic size of the polymer with length scale r^*. Use the above result to improve the free energy estimation by including the interaction between the monomers. We should have $F_{improved}(r, N, T) = U_{int}(r, N, T) + F_{entropic}(r, N, T)$. Find r^* as a function of v, a, N.

8. $r_{ideal} \sim N^{\frac{1}{2}}$, $r^* \sim N^{\alpha}$. Compare r_{ideal} and r^* at large N limit. Is $\alpha > \frac{1}{2}$, $\alpha < \frac{1}{2}$ or $\alpha = \frac{1}{2}$? Why is it a reasonable result?

Problem 5. *Partition function of H-atoms(10 points):* We can use what we have learned during statistical mechanics to understand some systems we have learned. One important system is the Hydrogen atom, the electron has discrete energy levels with $E_n = -\frac{E^*}{n^2}$. Here, $E^* = 13.6eV$ and n is the principle quantum number. Using Bohr model, we have the radius of atom to be $r_n = n^2 r_1$, $r_1 \approx 5 \times 10^{-11}m$. $k_B \approx 8.6 \times 10^{-5}eV/K = 1.38 \times 10^{-23}J/K$.

1. Write down the **formal expression** of partition function of the electron , Z_e, on a Hydrogen atom in the canonical ensemble at temperature T. It will be an infinite sum of series where

the principle quantum number $n \to \infty$. Will this partition function converge as $n \to \infty$? Why? (5 points)

2. The expression above requires some further physical arguments to estimate the probability for H-atoms to be at their ground states. What would you suggest to provide a physical sensible calculation and why? (5 points)

Classical Electrodynamics (2026 March)

Please use the SI unit system. If not, please indicate the system you use.

1. Explain the following terms. (6*5%)

(a) Lorentz gauge and Coulomb gauge

(b) Group and phase velocities

(c) Liénard-Wiechert potentials

(d) Complex Poynting's theorem

(e) Skin depth

(f) Brewster's angle

2. Derive the Green theorem step-by-step.

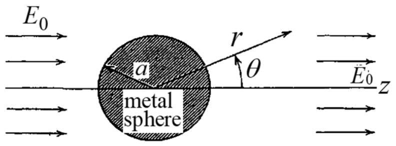
(a) Green's first identity and second identity. (10%)

(b) Application of the Green theorem to electrostatic boundary-value problems with the Green function satisfying $\nabla'^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ with $G_D(\mathbf{x},\mathbf{x}') = 0$ for $\mathbf{x}'$ on S. (10%)

3. A conducting sphere of radius a with net charge Q on its surface is placed in a uniform electric field $E_0\mathbf{e}_z$. Use the method of expansion to find

(a) the electric potential Φ outside the sphere; (10%)

(b) the surface charge density σ on the sphere. (10%)



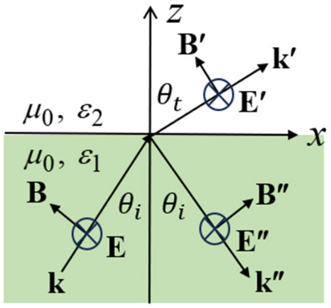
4. Derive the conservation of charge from the Maxwell equations. (10%)

5. For an incident wave with the electric field perpendicular to the plane of incidence (the x-z plane).

(a) Find E_0'/E_0 and E_0''/E_0. [Hint: Use the boundary conditions]. (14%)

(b) For normal incidence ($\theta_i = 0$), find E_0'/E_0 and E_0''/E_0. (6%)

[Hint: $\begin{cases} \mathbf{k} = k_x\mathbf{e}_x + k_z\mathbf{e}_z \\ \mathbf{k}' = k_x\mathbf{e}_x + k_z'\mathbf{e}_z \\ \mathbf{k}'' = k_x\mathbf{e}_x - k_z\mathbf{e}_z \end{cases}$ and $\begin{cases} \mathbf{E} = E_0\mathbf{e}_y \\ \mathbf{E}' = E_0'\mathbf{e}_y \\ \mathbf{E}'' = E_0''\mathbf{e}_y \end{cases}$]



Qualifying Exam for Quantum Mechanics (NTHU Physics, Spring 2026)

1. QM jargons (5 points each). Explain the following quantum mechanical terms:

- Bosons and Fermions
- Fermi golden rules
- Kramer degeneracy
- Canonical and kinetic momenta

Answer should be at least half page each, including detail explanations, equations and figures (if necessary).

2. Spin Angular Momentum (20 points). A system of two $s = 1/2$ spins are described by the effective Hamiltonian

$$H = J\, \vec{s}_1 \cdot \vec{s}_2 - B\, (s_{1z} + s_{2z})$$

Here $\vec{s}_1$ and $\vec{s}_2$ are the spin operators and J, B are constants. Find all eigenstates and energies of the effective Hamiltonian.

3. Perturbation Theory (20 points). Consider a perturbation $H' = \lambda x^4$ to a simple harmonic oscillator.

(a) Show that the first-order correction to the unperturbed eigenenergy is

$$E_n^{(1)} = \frac{3\lambda\hbar^2}{4m^2\omega^2}(1 + 2n + 2n^2)$$

(b) Estimate the quantum number n where the perturbation expansion starts to break down. Explain why this perturbation always fails for large enough n no matter how small the parameter λ is.

4. Born Approximation for 3D Scattering (20 points).

(a) Use the Born approximation to compute the differential cross section for the 3D scattering potential $V(r) = -V_0$ for $r \leq a$ and zero otherwise (here V_0 is positive).

(b) Show that, if the energy of the incident particle is low enough, the differential cross section becomes isotropic. You need to specify the criterion for the low-energy regime.

5. Rabi Oscillation (20 points). Consider a two-level system with $E_2 > E_1$ and the system is initially ($t = 0$) at the lower energy state, $c_1(0) = 1$ and $c_2(0) = 0$. A time-dependent potential is turned on for $t \geq 0$ with matrix elements given below:

$$V_{11} = 0 = V_{22}, \quad V_{12} = \gamma e^{i\omega t} = V_{21}^*$$

(a) Show by solving the Schrodinger equation directly that for $t \geq 0$

$$|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \sin^2\left(\sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_{21})^2}{4}}\, t\right)$$

Here $\omega_{21} = (E_2 - E_1)/\hbar$. The oscillation of the level population due to the periodic external field is called the Rabi oscillation.

(b) Use time-dependent perturbation theory to lowest nonvanishing order to compute the probability for the excited level ($E = E_2$) to be populated. When does the perturbative result fail? Explain what is missing within the perturbative picture that leads to its failure.