Quantum Mechanics Qualification
Fall, 2019

Problem 1 Answer the following questions *briefly*

(a) 12% Explain the following terms briefly: (i) collapse of state (ii) entanglement (iii) Aharonov-Bohm effect (iv) WKB approximation

(b) 4% What are generators for rotations?

(c) 4% What kind of particles does the Dirac equation describe?

(d) 5% Find the Hermitian conjugate of $3\frac{d}{dx}$ and $\hat{A} = x\frac{d}{dx}$ (in terms of $\hat{A}$).

(e) 5% Estimate the ground state energy (in units of eV) of a two-electron atom with the nuclear charge Z by using the uncertainty relation.

(f) 5% Consider 6 identical fermions in one dimension. Let x_i and p_i ($i = 1, 2, 3, ...6$) be the corresponding position and momentum operators for 5 particles. Which of the following operator(s) is(are) observable(s)?

$|p_1 - p_3| + |p_2 - p_4| + |p_3 - p_5|$, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$, $p_2^2 + p_4^2 + p_6^2$, $\sum_{i<j} \frac{e^2}{4\pi\epsilon_0 |x_i - x_j|}$.

Problem 2 Consider a perturbation $H' = \alpha x^4$ to the motion of the harmonic oscillator. Here α is a positive number. Let m and ω be the mass and the natural frequence of the oscillator. Answer the following questions:

(a) 10% Show that the first-order correction to the nth unperturbed eigen-energies are

$$E_n^1 = (3\hbar^2\alpha)/(4m\omega^2)[1 + 2n + 2n^2],]. \tag{1}$$

No matter how small α is, the perturbation expansion will break down for some large enough n. Why?

(b) 10% If we make the perturbation time-dependent, $H' = \alpha x^4 e^{-t^2/\tau^2}$, between $t = -\infty$ and $t = +\infty$. What is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = +\infty$? You might find the following identity useful. $\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \sqrt{\pi/a} e^{b^2/4a}$.

Problem 3 10% A particle of mass m is *confined* in a spherical cavity with radius R. This particle is otherwise free. Find the energy eigenvalues and corresponding normalized wavefunctions for states with angular momentum $l = 0$.

Problem 4 15%

(a) 15% Derive all spin eigenstates for two identical bosons of spin-1.

(b) 5% Consider a system of two particles with spins $s_1 = 1$ and $s_2 = 1/2$. By considering the addition of two spins, find all possible eigenvalues to the operator $(\mathbf{S}_1 - \mathbf{S}_2)^2$.

Problem 5 20%

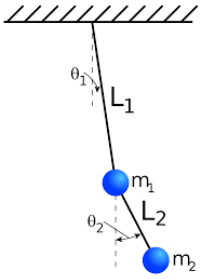
Consider the scattering between two spin-1/2 fermions. The fermions are identical and their massed are the same and is denoted by m. The interaction potential between two fermions can be approximated by $V_0(\vec{s}_1 \cdot \vec{s}_2)\frac{e^{-\alpha r}}{r}$, where $\vec{s}_1$ and $\vec{s}_2$ are the spin vector operators of two fermions and $V_0, \alpha > 0$. In the lab frame, the scattering is set up in the way that one fermion is initially at rest, while the other one is incident with a momentum $\hbar k$. Both fermions are not polarized. Use the Born approximation to calculate the differential cross section $\sigma(\theta, \phi)$ in the center of mass (CM) frame. What is the differential cross section in the lab frame?

Qualification examination (September 2019)

- Classical Dynamics -

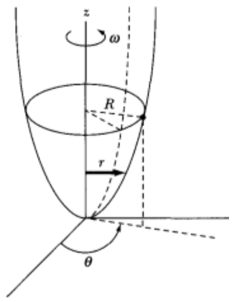
(20 points each, unless otherwise stated)

1. Mass m with velocity v moves horizontally toward an initially stationary sledge of incline angle θ and mass M. Assume the floor is smooth, but there is friction between m and M with dynamic friction coefficient μ_k. Find the maximum height m can climb on the sledge.

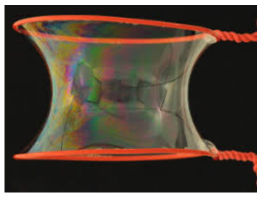


2. Derive the equation of motion for the double pendulum in Fig.2. Assuming $\theta_{1,2} \ll 1$, find the periods of this harmonic motion.
3. (a) (10 points) What is the gravitational potential both inside and outside a spherical shell of inner/outer radii b/a?

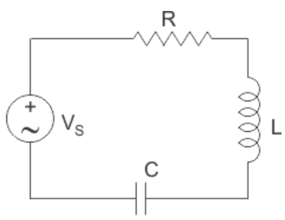
   (b) (Bonus 10 points) Find the gravitational self-energy of a sphere of mass M and radius R.

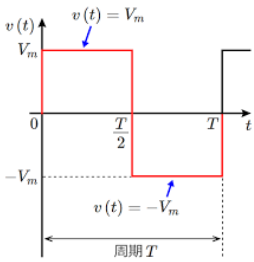


4. (10 points) A bead slides in Fig.3 along a smooth wire bent in the shape of $z = cr^2$. The bead rotates in a circle of radius R when the wire is rotating about its vertical symmetry axis with angular velocity ω. Find c.
5. A soap film is formed between two open circular wires of radii a, b and vertical separation h. Assume the amount of soap liquid is small so that gravity is not important. (a) Find the shape of soap film. (b) Does your derivations set an upper bound on h, as expect from experience. (c) How will your analysis change if the system is closed, i.e., the volume of air inside the soap bubble is fixed when we modify h.



6. Find the current I on an RLC-circuit that is driven by a square-wave AC voltage (see Figs.5 and 6.) Assume there is no current or charge at $t = 0$.





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Qualifying Examination – Statistical Mechanics

Fall, 2019

(20 points. Relativistic particles in a gravitational field)

Consider N relativistic ideal-gas particles at temperature T. The particles are confined to a 3D box of area A and $0 < z < \infty$. Because they are relativistic, the Hamiltonian is given by $H(\vec{p},\vec{q}) = \sum_{j=1}^{N} |\vec{p_j}|\, c + \sum_{j=1}^{N} mgz_j$.

1.1 Find the probability distribution for the height of a single particle: $P(z_1)$.

1.2 Find the average height of a single particle.

1.3 Find the probability distribution for the height and momentum of a single particle: $P(\vec{p_1}, z_1)$. (Hint: $\int_0^\infty x^2 e^{-ax} dx = 2a^{-3}$.)

1.4 Find the probability distribution for the energy of a single particle: $P(E)$, where $E = |\vec{p}|c + mgz$. (Hint: use a delta function to perform the transformation of a continuous probability distribution.)

2 (20 points. Ideal and non-ideal classical gas)

2.1 Find the Helmholtz free energy F for N ideal classical particles of individual mass m in volume V at temperature T.

2.2 Derive the state equation $PV = Nk_BT$ by $P = -\left.\frac{\partial F}{\partial V}\right|_{T,N}$.

2.3 For real particles, modify the free energy F as follows: (i) Real particles repel each other at close range with a hard volume b and (ii) they attract each other at long range. The total energy of attraction can be written as $-aN^2/V$, which should be added to the free energy. Show that these modifications lead us to the Van der Waals equation: $\left[P + a\left(\frac{N}{V}\right)^2\right](V - Nb) = Nk_BT$.

3 (20 points. Ising model) Consider the Ising model $H = -J\sum_{\langle i,j\rangle} \sigma_i\sigma_j$ where $\sigma_j = \pm 1$, and $\langle i,j\rangle$ denotes the nearest-neighbor pairs of sites.

3.1 For a system of coordinate number z, use mean-field approximation to find the critical temperature T_c below which spontaneous magnetization exists.

3.2 Show that the magnetic susceptibility $\chi \propto 1/(T - T_c)$ at $T \gg T_c$.

3.3 Use entropy argument to show that in 1D there is no phase transition, i.e., $T_c = 0$.

4 (20 points. 3D Ideal Fermi gas)

4.1 Consider a 3D ideal Fermi gas of spin S=5/2 at zero temperature, calculate the total internal energy by summing the electron energy all the way up to the Fermi energy.

4.2 Calculate the Fermi Pressure P.

4.3 Derive the Pauli magnetic susceptibility in the limit of zero magnetic field and low temperature.

5 (20 points. Bose-Einstein condensation) Consider a number conserved Bose gas with energy dispersion $\epsilon_p = C[\vec{\boldsymbol{p}}]^a$:

5.1 Show that there will be a Bose-Einstein condensation if $d > a$ where d is the dimension of the space.

5.2 Show that the critical temperature scales with the total number of particles, N, as $T_c \propto N^{a/d}$. (Hint: you do not need to do the full calculation, and may use dimensional argument for the density of states first.)

Electrodynamics Qualifying Examination, Sep., 2019.

You must provide the details or reasonings to justify your answers.

1. (5%+10%)
   (a) What are Green's first identity and Green's Theorem?
   (b) A point charge is located at $(0, 0, -d)$ below the grounded conducting plane at $z = 0$ which extends to infinite. Find the Green's function $G(\mathbf{x}, \mathbf{x}')$ which satisfies the Dirichlet boundary condition.

2. (10%) Show that $\vec{E} \cdot \vec{B}$ is Lorentz invariant.

3. (5%+5%)(a) What is gauge transformation?
   (b) Construct one set of the scalar and vector potentials in Coulomb gauge for a point charge q which is sitting at the origin in a uniform $\vec{B} = B_0\hat{x}$.

4. (5%+10%) (a) What is Maxwell's stress tensor?
   (b) Use Maxwell's stress tensor to calculate the force experienced by a point charge q in a uniform electric field $\vec{E} = E_0\hat{y}$.

5. (15%) A thin-walled metal sphere (radius $= a$) bears a net positive charge Q. The permittivity (dielectric constant) of the entire region surrounding the metal sphere is $\epsilon = \epsilon_0(1 - b^2/r^2)$, where b is the positive constant and $a > b$. In terms of the given quantities and appropriate numerical and physical constants, calculate the electrostatic energy of this distribution.

6. (15%) A uniform disk of mass m and radius R has a charge Q distributed uniformly alone its perimeter. It is set spinning about one of its diameters with constant angular frequency ω. Given that its moment of inertia is $mR^2/4$, find the $g-$factor, defined as the ratio μ/L between magnetic moment and angular momentum.

7. (10+10%)Assume that plane electromagnetic waves are incident on the boundary between two isotropic dielectric media.
   (a) Use the appropriate boundary conditions to derive Snell's law.
   (b) Use the appropriate boundary conditions to derive an expression for the incident angle for which the maximum polarization occurs.