Qualifying Examination – Classical Mechanics

Spring, 2018

1. (Brachistochrone problem, 16 points)

A classic problem of the calculus of variations involves finding the quickest path for a particle to move under gravity from a point (x_1, y_1) to some lower point (x_2, y_2).

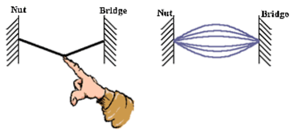
(a) Find the expression of such a path (called cycloid, 擺線).

(b) It is rumored that the actual time of travel is independent of where you release the particle. In other words, no matter how close (x_1, y_1) is to (x_2, y_2), it always takes the same time to reach the latter. Check if this claim is true.

2. (Catenary 懸鏈線, 8 points)

Find the curve of an idealized hanging chain under its own weight when supported only at its ends.

3. (Vibrating strings, 8 points)



A guitar string of length *L* and mass *M* is stretched under tension *T*. If pluck at the middle to a height of *h* and release without initial velocity, determine the subsequent displacement of the string at time *t*.

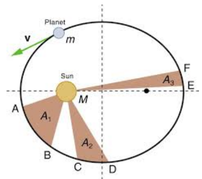
4. (Damped harmonic oscillator, 24 points)

(a) Solve the damped harmonic motion, $m\ddot{x} = -kx - \alpha\dot{x}$ with $\alpha > 0$, for the under- and critically-damped cases, respectively.

(b) With the addition of an external drive, $m\ddot{x} = -kx - \alpha\dot{x} + F_0 \sin\Omega t$. Find the steady-state solution for the under-damped case.

(c) Calculate the resonant frequency at which the amplitude is a maximum.

5. (Central-force motion, 20 points)



Show that planets of mass m and energy E orbiting a mass-M star follow different orbits according to $E > 0$ (hyperbola), $E = 0$ (parabola), $V_{mini} < E < 0$ (ellipse), and $E = V_{mini}$ (circle) where $V(r) \equiv -\frac{GMm}{r} + \frac{\ell^2}{2\mu r^2}$, $\mu \equiv \frac{Mm}{M+m}$, and ℓ is the angular momentum.

6. (Motion in a non-inertial reference frame, 16 points)

(a) One myth of Coriolis effect is that it causes water in your toilet bowl to spin in certain chirality[1]. Please derive the Coriolis force.

(b) Calculate the deflection of a falling particle on latitude λ in the northern hemisphere due to the Coriolis force. Use h to denote the initial height.

7. (Kinematics of two-particle collisions, 8 points)

Calculate the differential and total cross sections for the elastic scattering of a particle from an impenetrable sphere of radius *a*.

[1] In reality, the shape of bowl and the angle at which liquid initially enters the bowl are the true culprit.

Qualifying Examination – Statistical Mechanics

Spring, 2018

1. (Ideal gas, 15 points) For N ideal classical atoms in volume V and at T,
   (a) Find their entropy S and chemical potential μ within canonical ensembles. Use them to prove $TS = U + PV - \mu N$.
   (b) Show that the total entropy increases (remains unchanged) when two different (identical) such gases are mixed.
   (c) Derive the familiar $PV = Nk_BT$ by use of grand canonical ensembles.
2. (Fermi gas, 15 points) Find the expressions for (a) chemical potential and (b) specific heat at $T \ll$ Fermi energy ε_F, and (c) quantum pressure at T=0 for an ideal nonrelativistic Fermi gas in 3-D.
3. (Relativistic electrons, 5 points) For electrons with an energy $\varepsilon \gg mc^2$ where m is the rest mass, the energy is given by $\varepsilon \approx pc$ where p is the momentum. Show that the total energy of the gas at ground state obeys $U_0 = 3N\varepsilon_F/4$.
4. (Bose gas, 10 points) (a) Find the Bose-Einstein condensation temperature, T_{BE}, for an ideal Bose gas in 3-D. (b) Explain why μ has to remain zero at $T \leq T_{BE}$.
5. (Thermal radiation, 10 points) Please derive Stefan's T^4-law. Use it to estimate the amount of energy your body loses each day in the form of thermal radiation.
6. (Ising model, 10 points) Ising Hamiltonian obeys $H = -J\sum_{\langle i,j\rangle} S_i S_j$ where $\langle i,j\rangle$ denotes the nearest neighbors and $S = \pm 1$. The number of nearest neighbors is q.
   (a) Find the mean-field Curie temperature T_C below which spins start to aligned.
   (b) Show that the magnetic susceptibility $\chi \propto 1/(T - T_c)$ at $T \gg T_c$.
7. (Negative temperature[1], 15 points) Interaction energy of dipole moment $2\mu_{\mathrm{B}}S_{\mathrm{z}}$ in a magnetic field B equals $-2\mu_{\mathrm{B}}S_{\mathrm{z}}B$ where $\mu_{\mathrm{B}} \equiv \frac{e\hbar}{2mc}$ is the Bohr magneton.
   (a) Find the entropy S and internal energy U for N spins $S_{\mathrm{z}} = \pm 1/2$ at T and B.
   (b) Use (a) to draw S vs. U. Then plot T vs. U by use of $\frac{1}{T} \equiv \frac{dS}{dQ} = \frac{dS}{dU}$.
   (c) Notice that negative T appears on half of your plot. How to make sense of it? What to expect if a T>0 system gets in thermal contact with another T<0 one?
8. (Biased random walk, 20 points) A 1-D drunkard, evicted from the pub at x=0, has a higher probability p>0.5 of moving forward. Assume same stride ℓ.
   (a) Write down the probability $P(x, N)$ of finding him at $x = n\ell$ after N steps.
   (b) Use $\lim_{n\gg 1} n! \sim n^n e^{-n}\sqrt{2\pi n}$ to reduce (a) to Gaussian form when $\left|\frac{x}{\ell}\right| \ll N$.
   (c) Find the expectation values, $\langle x\rangle$ and $\langle x^2\rangle$. Argue if these results are intuitive.
   (d) What will an impenetrable wall at $x = -x_0 < 0$ change the result in (b)?

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[1] This is not a fictional idea. An article titled "Relaxation to Negative Temperatures in Double Domain Systems" just appears in https://journals.aps.org/prl/pdf/10.1103/PhysRevLett.120.060403

Ph.D. Qualifying Exam: **Classical Electrodynamics**

March 2018

1. (30%) Explain the following terms qualitatively and quantitatively.

(a) Poynting theorem (5%)

(b) Dirichlet and Neumann boundary conditions (5%)

(c) Helmholtz wave equation (5%)

(d) Synchrotron radiation (5%)

(e) Kramers-Kronig relations (5%)

(f) Lorentz gauge and Coulomb gauge (5%)

2. (20%) Boundary conditions and applications.

(a) $\nabla \cdot \mathbf{D} = \rho_f$. Find the boundary condition for the normal component of **D**, $D^{\perp}$. (6%)

(b) $\nabla \times \mathbf{H} = \mathbf{J}_f$. Find the boundary condition for the tangential component of **H**, $\mathbf{H}^{//}$. (6%)

(c) Consider the interface between two dielectric materials with ε_1 and ε_2 as shown in the figure. Find the relations between the normal and the tangential components of the electric fields. Assume that there is no surface charge, i.e., $\sigma_f = 0$. (8%)

$$\varepsilon_1,\ E_1^{\perp},\ E_1^{//}$$

$$\varepsilon_2,\ E_2^{\perp},\ E_2^{//}$$

3. (20%) Green function

(a) What are Green's first identity and Green's theorem? (10%)

(b) For a point charge **outside** a grounded conducting sphere, find the Green function $G(\mathbf{x}, \mathbf{x}')$ that satisfies Dirichlet boundary condition. (10%) [Hint: the method of images.]

4. If $\mathbf{E}$ and $\mathbf{B}$ are perpendicular in the laboratory and $|E| = 2|B|$, find a reference frame such that the field is pure electric or magnetic? If yes, what is the velocity of the reference frame relative to the laboratory? (10%)

[Hint: Gaussian unit $\mathbf{E}'_{\|} = \mathbf{E}_{\|}$, $\mathbf{E}'_{\perp} = \gamma_0 \left(\mathbf{E}_{\perp} + \frac{\mathbf{v}_0}{c} \times \mathbf{B}_{\perp} \right)$ and $\mathbf{B}'_{\|} = \mathbf{B}_{\|}$, $\mathbf{B}'_{\perp} = \gamma_0 \left(\mathbf{B}_{\perp} - \frac{\mathbf{v}_0}{c} \times \mathbf{E}_{\perp} \right)$]

5. (20%)

(a) Starting from the Maxwell equations, derive the dispersion relation (i.e., the relation between the wave frequency ω and the propagation constant k) for a plane electromagnetic wave in an infinite and uniform medium of conductivity σ, electrical permittivity ε, and magnetic permeability μ.

(b) Assume that the medium is a good conductor, derive an expression for its skin depth δ.

[Hint: $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$]

Qualifier of Quantum Mechanics (Spring 2018)

1. (5+5+5+5 pts) Briefly answer the following questions:
   (a) What is the consequence for a system being translational invariance in quantum mechanics, why ?
   (b) What is WKB method ?
   (c) What is CG coefficient ?
   (d) What is spontaneous emission?

2. (5+5+10 pts) Consider two particles of the same mass *m* in one dimension. Two particles are connected by a spring with spring constant *k*.
   (a) Assuming the two particles are non-identical, calculate the eigen-energies of such a system with a total momentum p.
   (b) Assuming the two particles are identical bosons, calculate the eigen-energies of such a system with a total momentum p.
   (c) Suppose that two particles carry charges of q and $-q$ and move along *x* axis. A uniform electric field *E* is applied along + *x* direction. Assuming the Coulomb interaction between these two charges is negligible, find the root-mean-square relative distance (i.e., minimum of $\sqrt{\langle(x_1-x_2)^2\rangle}$) between two particles when in the nth eigenstate..

3. (10pts) Consider the neutron-neutron scattering where the interaction potential is approximated by $V(r)=V_0\vec{s_1}\cdot\vec{s_2}e^{-r/a}$, where $\vec{s_1}$ and $\vec{s_2}$ are the spin vector operators of the two neutrons (spin 1/2), and $V_0>0$. Find the differential cross section in the first Born approximation for s-wave scattering.

4. (10+5+10 pts)Consider a perturbation $H'=\alpha x^4$ to the harmonic oscillator problem.
   (a) Work out the first-order correction to the eigen-energies of state $|n^{(0)}\rangle$. (hint: $\mathrm{a}=(ip+m\omega x)/\sqrt{2\hbar m\omega}$ where ω is the natural frequency.)
   (b) No matter how small α is, the perturbation expansion will break down for some large enough *n*. Why?
   (c) If we make the perturbation time-dependent, $H'=\alpha x^4 e^{-t^2/\tau^2}$, between $\mathrm{t}=-\infty$ and $\mathrm{t}=+\infty$. What is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $\mathrm{t}=+\infty$?
   You might find the following identity useful.
   $$\int_{-\infty}^{+\infty} dx\, e^{-ax^2+bx}=\sqrt{\frac{\pi}{a}}exp\frac{b^2}{4a}$$

5. (10 pts) Work out the Clebsch-Gordan (CG) coefficients of
   $$1\otimes 1/2=1/2\oplus 3/2$$

6. (10+5pts) (a) Express the following matrix $H = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, in terms of a linear combination of Pauli matrices and Identity matrix $\boldsymbol{I}$. (b) If we can define an effective "magnetic field" for an effective spin (iso-spin) system, such that $H = -\boldsymbol{B} \cdot \boldsymbol{S}$ what will be the effective magnetic field, **B** ?