Quantum Mechanics Qualification
Fall, 2017

Problem 1 Answer the following questions *briefly*

(a) 20% Explain the following terms briefly: (i) collapse of state (ii) Aharonov-Bohm effect (iii) Clebsch-Gordan coefficients (iv) entangled states

(b) 5% Express the Hermitian conjugate of $\hat{O}$ in terms of $\hat{O}$ with $\hat{O}$ being (i) $2x\frac{d}{dx}$ and (ii) $[A, B]$, where A and B are observables.

(c) 5% Consider a particle whose position is denoted by $\vec{r} = (x, y, z)$ and total angular momentum is denoted by $\vec{J}$. Find $\exp(iJ_y\phi/\hbar)\, x^2 \exp(-iJ_y\phi/\hbar)$ in terms of x and ϕ.

Problem 2 Consider a particle of mass m that moves freely in one dimension. Suppose that at $t = 0$, the wavefunction of the particle has the properties: $\Delta x = a$, $\Delta p = \frac{\hbar}{2a}$, $\langle x \rangle = x_0$, $\langle p \rangle = p_0$.

(a) 5% What is the general form of the wavefunction at $t = 0$?

(b) 7% Find the position operator in the Heisenberg picture $\hat{\mathrm{x}}_H(t)$ (for $t > 0$) in terms of the operators $\hat{\mathrm{x}}$ and $\hat{\mathrm{p}}$ defined in the Schrodinger's picture. Calculate the commutator $[\hat{\mathrm{x}}_H(t), \hat{\mathrm{x}}_H(t')]$

(b) 8% Consider a special case when $p_0 = 0$, by solving appropriate equations of motions, find $\Delta x(t)$ and $\Delta p(t)$ for $t > 0$.

Problem 3 10% Consider two particles of the same mass m in one dimension. Two particles are connected by a spring with spring constant k. Suppose that the total momentum of two particles be p, find all possible total energies for the following cases: (i) two particles are non-identical (ii) two particles are identical fermions (iii) two particles are identical bosons.

Problem 4 10% Work out the addition of two angular momenta $l = 1$ and $s = 1/2$, i.e., if we denote the normalized eigenstates for $l = 1$ as Y_1^m and those for $s = 1/2$ as $|\uparrow\rangle$ and $|\downarrow\rangle$, find the normalized eigenstate $|J, m\rangle$ in terms of these states.

Problem 5 Consider a perturbation $H' = \alpha x^4$ to the motion of the harmonic oscillator.

Here α is a positive number. Let m and ω be the mass and the natural frequence of the oscillator. Answer the following questions:

(a) 10% Show that the first-order correction to the nth unperturbed eigen-energies are

$$E_n^1 = (3\hbar^2\alpha)/(4m\omega^2)[1 + 2n + 2n^2],]. \tag{1}$$

No matter how small α is, the perturbation expansion will break down for some large enough n. Why?

(b) 10% If we make the perturbation time-dependent, $H' = \alpha x^4 e^{-t^2/\tau^2}$, between $t = -\infty$ and $t = +\infty$. What is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = +\infty$? You might find the following identity useful. $\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \sqrt{\pi/a} e^{b^2/4a}$.

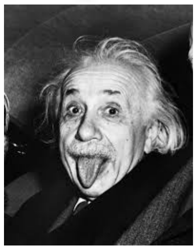
Problem 6 10% Consider the mutual elastic scattering of two particles. The Hamiltonian for this system is $H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(\vec{r}_1 - \vec{r}_2)$, where $V(\vec{r}) = g\delta^3(\vec{r})$. In the lab frame, the scattering is set up in the way that one particle is initially at rest, while the other one is incident with a momentum $\hbar k$. Use the Born approximation to calculate the differential cross section $\sigma(\theta, \phi)$ in the center of mass (CM) frame. What is the differential cross section in the lab frame?

Qualifying Examination – Statistical Mechanics

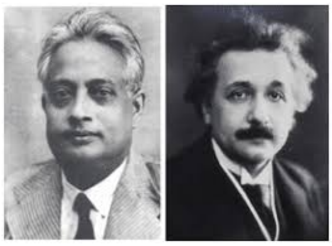
Fall, 2017

1. (Irreversible processes and fluctuations, 20 points + 5 bonus points) Please explain the physics behind the following phenomena in Statistical Mechanics:
   (a) How ink **diffuses** in water (Fig.1).

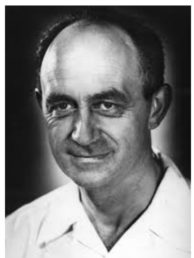
   (b) The **drag** experienced by a hard sphere of radius R that moves through a liquid of **viscosity** η - the Stoke's law.
   (c) The magnitudes of **mobility**, $\mu \equiv \frac{v_{drift}}{F_{external}}$, and diffusion coefficient, D, naturally vary with different gases, but their ratio turns out to be universally $\frac{1}{k_B T}$ – **Einstein** (Fig.2) **model**.



   (d) Relation between (macroscopic) dissipation and the (microscopic) fluctuating Brownian force - **fluctuation-dissipation theorem**.
   (e) (Bonus, 5 points) Tell me what you know about the **Langevin equation**, **Markoff processes**, and **Fokker-Plank equation**.
2. (Ideal gas, 30 points + 5 bonus points) For N ideal particles of mass m in 3-D volume V,



   Case 1: classical particles
   (a) Find the expression of entropy at temperature T.
   Case 2: bosons
   (b) Find the Bose-Einstein (Fig.3) condensation temperature, T_{BE}
   (c) Find the specific heat, C_V, at $T < T_{\mathrm{BE}}$.
   Case 3: fermions



   (d) Find the Fermi (Fig.4) energy, E_F.
   (e) Find the degenerate pressure at T=0 in the non-relativistic limit.
   (f) (Bonus, 5 points) Find C_V at $T \ll E_F$.
   Case 4: photons - massless bosons
   (g) Find the energy radiated by a black body per unit time and surface area.
3. (Ising model, 28 points) The Ising (Fig.6) Hamiltonian is $H = -J\sum_{\langle i,j\rangle} S_i S_j$ where $\langle i,j\rangle$ denotes the nearest neighbors and $S = \pm 1$. Denote the number of nearest neighbors for each S_i by q

   Case 1: Ising chain: exact solution
   (a) Solve the Ising model in one dimension exactly and show that $T_C = 0$, i.e., there is no phase transition to a magnetic state at any T for an Ising chain.

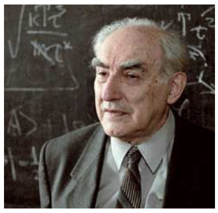
Case 2: mean-field approximation

(b) Find the Curie[1] (Fig.7) temperature T_C below which all spins are parallel aligned.

(c) Show that the magnetic susceptibility $\chi \propto \frac{1}{T\text{-}T_\mathrm{C}}$ at $T \gg T_\mathrm{C}$.

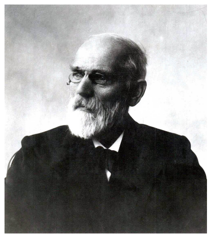
Case 3: Ginzburg-Landau (Fig.8,9) free energy

(d) Show that magnetization $m \propto \sqrt{T_\mathrm{C}\text{-}T}$ for $T \rightarrow T_\mathrm{C}^-$



4. (Imperfect classical gas, 12 points + 5 bonus points)

(a) By use of the facts that real particles (i) repel each other at close range with a hard volume v and (ii) attract each other at long range and so add a potential energy term[2] $-\alpha\left(\frac{N}{V}\right)^2 V$, derive the Van der Waals (Fig.10) equation of state: $\left[P + \alpha\left(\frac{N}{V}\right)^2\right](V - Nv) = Nk_BT$ from the simpler case of an ideal gas.



(b) (Bonus, 5 points) Determine the critical temperature at which there is no separation between the vapor and liquid phases. Why does the Van der Waals equation of state need to be modified below this critical temperature and how?

5. (Crumpled thin sheet, 10 points)

Creases of different length ℓ will appear on a crumpled paper (Fig.11). Imagine the first crease to be comparable to the paper size, R. As crumpling proceeds, it gets decimated as $\ell = R\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{3}{5}\cdot\frac{2}{5}\cdots$ in an idealized case. A little shuffling gives $\ln\frac{R}{\ell} = N_1\ln\frac{5}{2} + N_2\ln\frac{5}{3}$ where $N_1(N_2)$ denote the number of times when the crease is shortened by $\frac{2}{5}\left(\frac{3}{5}\right)$ and $N_1 + N_2{=}N$ depends on the severity of crumpling. Setting $\ln\frac{5}{2} = \alpha + \beta$ and $\ln\frac{5}{3} = \alpha\text{-}\beta$, we get $\ln\frac{R}{\ell} = (N_1 + N_2)\,\alpha + (N_1\text{-}N_2)\,\beta$ that resembles a random walk in a wind of strength α that sways the drunkard throughout his walk. Find the probability distribution of ℓ.

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[1] Pierre Curie (1859 – 1906) received the Nobel Prize in Physics with his wife, Marie Curie, in 1903.

[2] The parameter, α, describes the strength of pairwise attractive force between particles, while the joint probability gives $\left(\frac{N}{V}\right)^2$, and V comes from integrating the energy density $-\alpha\left(\frac{N}{V}\right)^2$ over the entire volume.

Ph.D. Qualifying Exam: **Classical Electrodynamics**

Sep. 2017

1. (30%) Explain the following terms qualitatively and quantitatively.

(a) Skin depth (5%)

(b) Poynting theorem (5%)

(c) Plasma frequency (5%)

(d) Synchrotron radiation (5%)

(e) Retarded Green function (5%)

(f) Lorentz gauge and Coulomb gauge (5%)

2. (20%) Two concentric conducting shells of inner and outer radii a and b (b>a), respectively. The inner shell is connected to a potential $V(a,\theta)$ (to be given), while the outer shell is grounded $V(b,\theta)=0$.

(a) If $V(a,\theta)=V_0$ (constant), find the potential at $r<a$, $a<r<b$, and $r>b$. (10%)

(b) If $V(a,\theta)=V_0(1-\cos^2\theta)$, find the potential everywhere between the shells ($a<r<b$). (10%) [Hint: use Legendre polynomials $P_0(x)=1$, $P_1(x)=x$, and $P_2(x)=(3x^2-1)/2$].

3. (10%) Consider an infinite long hollowed iron bar of inner and outer radii a and b, respectively, carries a uniform magnetization, $\mathbf{M}=M_0\hat{z}$.

(a) Find the bound surface currents $\mathbf{K}_b$ on the inner and outer surfaces. (5%)

(b) Find the magnetic field at the central axis. (5%) [Hint: use cylindrical coordinate, (ρ,ϕ,z).]

4. (20%) Green function

(a) What are Green's first identity and Green's theorem? (10%)

(b) For a point charge outside a grounded conducting sphere, find the Green function $G(\mathbf{x},\mathbf{x}')$ that satisfies Dirichlet boundary condition. (10%) [Hint: the method of images.]

5. (20%) A hollow rectangular waveguide has a cross section of $a\times b=2.54\ \text{mm}\times 1.27\ \text{mm}$.

(a) Estimate the cutoff frequencies for the first three modes (TE_{10}, TE_{20}, TE_{01}). (10%)

(b) Qualitatively plot the dispersion relation ($\omega-k_z$ diagram) of the dominant TE_{10} mode. (10%)

Qualification Exam for PhD Candidates (Classical Mechanics, Sept. 2017)

1. (10%) What is the total cross section for the elastic scattering of a beam of particles of radius r from a fixed solid sphere with radius R?

2. (15%) An incident proton with relativistic energy E hits a proton with mass m at rest, producing a proton-antiproton pair:

   $$p + p \to p + p + (p + \bar{p}).$$

   Use special relativity to calculate the minimum energy of the incident proton.

3. (15%) Consider the motion of a particle with mass m in a central force field and use the spherical polar coordinates (r, θ, ϕ). The potential energy is $V(r)$. (a) Write down the Lagrangian of the system. (b) Derive the Hamiltonian from the Lagrangian. (c) Write down the Hamilton's equations. (Do not solve the equations).

4. (15%) A bucket of water is set spinning about its symmetry axis with constant angular velocity ω. Determine the shape of the surface of water in the bucket.

5. (15%) Consider two coupled oscillators with the same mass m and spring constant k. The potential energy is

   $$V = k(x_1^2 + x_2^2)/2 + \epsilon x_1 x_2$$

   where ϵ is the coupling constant and x_i are the coordinates of the oscillators. Find the angular frequencies of vibration.

6. (15%) Write down (without proof) the most general solution of the wave equation

   $$\frac{\partial^2 F(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F(x,t)}{\partial t^2} = 0$$

   and explain the physical meaning of the solution.

7. (15%) A particle with mass m and total energy E moves in one dimension. The potential energy is $V(x) = C|x|$ where C is a positive constant. Using action-angle variables to determine the period of the motion.