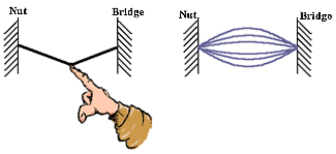
Qualifying Examination – Classical Mechanics

Fall, 2018

1. (Gravitation, 10 points) Find the gravitational potential at $r > a$, $b < r < a$, and $r < b$ for a spherical shell of inner radius b, outer radius a, and mass density ρ?
2. (10 points) If a particle of mass m falls from rest under gravity and experiences a resisting force $-kmv$, find the time lapse when it accelerates from v_0 to v_1?
3. (Calculus of variations, Example 6.6 in Marion's, 10 points)
   Find the curve $y(x)$ of length ℓ bounded by the x-axis on the bottom that passes through the points $(\pm a, 0)$ and encloses the largest area..
4. Use the Lagrangian mechanics to solve the following two problems.
   (a) (Example 7.6 in Marion's, 10 points) Find the frequency of small oscillations of a simple pendulum placed in a railroad car that has a constant acceleration a in the x-direction.
   (b) (Example 7.10 in Marion's, 10 points) A particle of mass m starts at rest on top of a smooth fixed hemisphere of radius a. Find the force of constraint, and determine the angle at which the particle leaves the hemisphere.
5. (Vibrating strings, 15 points) A string of length L and mass M is taut under tension T. If pluck at the middle to a height of h and release without initial velocity, find the subsequent displacement of string at time t.



6. (Damped harmonic oscillator, 10 points) Find the general solution for an under-damped oscillation under external drive obeys $m\ddot{x} = -kx - \alpha\dot{x} + F_0\cos\omega t$
7. (Central-force motion, 15 points) Show that planets of mass m and energy E orbiting a mass-M star follow different orbits according to $E > 0$ (hyperbola), $E = 0$ (parabola), $V_{\text{mini}} < E < 0$ (ellipse), and $E = V_{\text{mini}}$ (circle) where V_{mini} is the minimum value of effective potential energy, $V(r) \equiv -\frac{GMm}{r} + \frac{\ell^2}{2\mu r^2}$, $\mu \equiv \frac{Mm}{M+m}$, and ℓ denotes the angular momentum.
8. (Dynamics of a system of particles, Prob.9-50 in Marion's, 10 points)
   A fixed force center scatters a particle of mass m according to the force law $F(r) = k/r^3$. If the initial velocity of the particle is u_0, show that the differential scattering cross section is $\sigma(\theta) = \frac{k\pi^2(\pi-\theta)}{mu_0^2\theta^2(2\pi-\theta)^2\sin\theta}$

Qualifier of Quantum Mechanics (Fall 2018)

1. (5+5+5+5 pts) Briefly “explain briefly” the meaning of following question:
   (a) What is rotational symmetry and its implication to a quantum system ?
   (b) What is Aharonov-Bohm effect ?
   (c) What are fermions and bosons ?
   (d) What is Fermi-Golden rule ?

2. (10+5 pts) Consider a spin 1/2 electron in the following situation: (a) One first measure the spin along the z-axis using a Stern-Gerlach apparatus, and define the state to be a state of eigenvalue, $□/2$. What is the probability that a subsequent measurement of the spin in the direction $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ still yields $+□/2$?
(b) After the measurement of the electron’s spin along the axis $\hat{n}$ with the eigenvalue $+□/2$, what is the probability that a subsequent measurement of the spin along the x-axis yields $+□/2$?

3. (10+5 pts) (a) Consider a 1D attractive potential, $V(\mathrm{x}) = -V_0$ for $|x| < a$. Show that there must be at least one bound state for any value $V_0 > 0$ by using variational method. (b) Explain why this method does not apply in 3D case (i.e. in 3D, V_0 has to be larger than a certain value in order to have a bound state possible.

3. (10+10 pts) A point particle of mass m and incident energy E is scattering off the potential $V(\vec{r}) = ge^{-r^2/R^2}$. Calculate the first Born approximation to the differential and total cross sections.

4. (10pts) For 87 Rb atom, the nuclear spin is I=3/2 and has one electron in the s-state with spin S=1/2. The Hyperfine coupling is like $H = A\boldsymbol{I}\cdot\boldsymbol{S}$, where I and S are nuclear spin and electron spin operators r espectively. Calculate the hyperfine splitting energy.

5. (10+5+5 pts) Consider a one-dimensional harmonic oscillator with mass m and natural frequence ω. The oscillator is in its ground state $|0\rangle$ at $\mathrm{t} = -\infty$. A time-dependent potential is switched on as follows

$$V(x,t) = \frac{V_0 x}{1 + (t/\tau)^2}$$

where τ is a positive constant and V_0 is also a constant.
(a) Find the probability that the oscillator is in the state $|n\rangle$ $(\mathrm{n} \neq 0)$ at $\mathrm{t} = \infty$ to the order of ${V_0}^2$ (Hint: Using the first order time-dependent perturbation).
(b) Suppose that for $\mathrm{t} \geq 0$, $V(x,t) = V_0 x$. Find the ground state wavefunction ψ for the new Hamiltonian.
(c) What’s the probability for the initial ground state still stays in the new ground state ?

Qualifying Examination – Statistical Mechanics

Fall, 2018

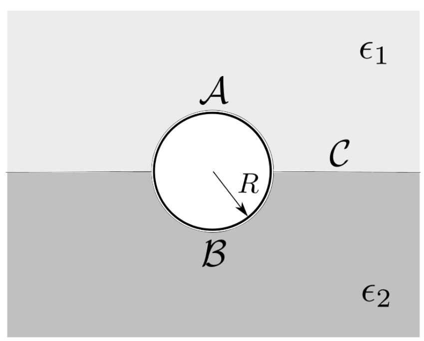
1. (Two level systems, 20 points) Consider N impurity atoms trapped in a solid matrix. Each impurity can be in one of two states, with energy 0 and ϵ, respectively. The total energy of the system is E. In the micro-canonical ensemble, (a) Find the $\Omega(E, N)$. (You don't need to worry about the number of excited state not being an integer.) (b) Find the entropy $S(E, N)$. (Use Sitrling's formula to simplify.) (c) The equilibrium temperature $T = T(E, N, \epsilon)$. (d) The heat capacity of the system and its low/high temperature expansions.
2. (Fermions in a two-level system, 15 points) Consider a system of N independent fermions. Assume that the single-particle Hamiltonian have only two energy levels, with energy 0 and energy ϵ. However, the two levels have degeneracies n_0 and n_1, which are both integers. (a) For the case of $n_0 = n_1 = 3$, with $N = 3$. Find the chemical potential μ, as a function of temperature. What is the Fermi energy $\epsilon_F = \mu(T = 0)$? (b) For the case of arbitrary value of n_0 and n_1, but with $N = n_0$. Find the chemical potential μ, as a function of temperature at low temperature limit. What is the Fermi energy $\epsilon_f = \mu(T = 0)$?
3. (Simple harmonic oscillator, 15 points) Consider a 1D quantum simple harmonic oscillator with ground state energy $\epsilon_0 = \hbar\omega/2$. In thermal equilibrium: (a) Find the canonical partition function. (b) Find the free energy F. (c) Find the average energy U from F. (d) Find the average number of excitation $\langle n \rangle$.
4. (2D ideal Bose/Fermi gases, 20 points) Consider an ideal Bose/Fermi gas in 2D. The N particles in the gas each have mass m and are confined to a box of $L \times L$. (a) Calculate the density of state for Bose gas $D_{BE}(\epsilon)$ and Fermi gas $D_{FD}(\epsilon)$ respectively. (b) Calculate the Einstein temperature in terms of the given parameters. (c) Calculate the Fermi energy as a function of the particle density. (d) Calculate the average particle number as a function of μ exactly.
5. (Ising model, 30 points) Consider the Ising model $H = -J\sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_j \sigma_j$ where $\sigma_j = \pm 1$, h is the external field, and $\langle i, j \rangle$ denotes the nearest-neighbor pairs of sites. (a) Use mean-field approximation to derive the self-consistent equation for the magnetization $m = m_j = \langle \sigma_j \rangle$ for system with coordination number z. (b) For the case of $h = 0$, identify the critical temperature T_c below which spontaneous magnetization exists. (c) For 1D lattice with periodic boundary condition $\sigma_{N+1} = \sigma_1$, show that the canonical partition function can be expressed as $Z = \sum_{\{\sigma\}} \prod_j T(\sigma, \sigma_{j+1})$, where $T(\sigma_j, \sigma_{j+1})$ is the transfer matrix. Find explicitly the matrix elements of the transfer matrix. (You might want to write H in a more symmetric way.) (d) Show that in the thermodynamic limit the free energy per site is $F/N = -k_B T \ln(\lambda_+)$ where λ_+ is the largest eigenvalue of the transfer matrix. (e) Find the T_c for this case.

Electrodynamics Qualifying Examination, Sep., 2018.

You must provide the details or reasonings to justify your answers.

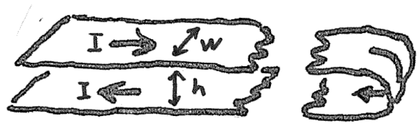
1. (5% each) Concisely explain the following terms:
   (a) Linear and circular polarization of light.
   (b) Perfect conductor and superconductor, how they differ from each other.
   (c) Skin depth.
   (d) Synchrotron radiation.
   (e) Why the sky is blue in the noon of a sunny day and red at sunset.

2. (5+5+5+5%) The center of a conducting sphere of radius R is located on the flat boundary between two dielectrics each filling half of the whole space outside the sphere. The dielectric permittivities are ϵ_1 and ϵ_2. The conducting sphere is held at potential V. Consider the space outside of the conducting sphere.



   (a) Show that the potential $\Phi = VR/r$ satisfies the required boundary conditions on the plane $\mathcal{C}$ separating dielectrics as well as on the sphere.
   (b) Find the free charge density σ on the surface of the conducting sphere and the total amount of free charge Q on it.
   (c) Find the bound charge densities σ_b on the spherical boundaries $\mathcal{A}$ and $\mathcal{B}$ of the dielectrics.
   (d) Find the bound charge density σ_b on the flat boundary $\mathcal{C}$ between the dielectrics.

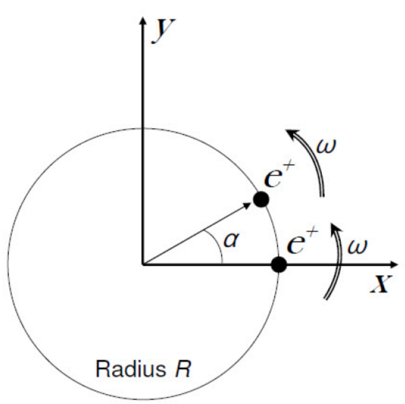
3. (5+5+5%) A transmission line is constructed from two thin metal "ribbons", of width w, a very small distance $h \ll w$ apart. The current travels down one trip and back along the other one. In each case, it spreads out uniformly over the surface of the ribbons.

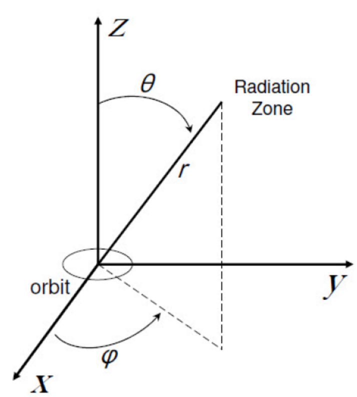


(a) Find the capacitance per unit length, $\mathcal{C}$.
(b) Find the inductance per unit length, $\mathcal{L}$.
(c) What is the product $\mathcal{LC}$, numerically? What does it mean? (Also specify the unit carefully.)

4. (10%) Suppose the yz plane forms the boundary between two linear media with $\mu = \mu_0, \epsilon_1$ for $x < 0$ and $\mu = \mu_0, \epsilon_2$ for $x > 0$. A plane wave of frequency ω, traveling in the x direction and polarized in the y direction, approaches the interface from $x = -\infty$. Derive the transmission coefficient and express it in terms of the reflection indices n_1 and n_2.

5. (5+5+5%) Tow point particles(each having the same electric charge $+e$) travel in the xy-plane around the circumference of a circle with radius R. Both charges travel at the same constant angular velocity ω but maintaining a fixed angular separation α throughout the motion. Assuming that the motion of the particles is non-relativistic ($\omega R \ll c$).
(a) Find the electric and magnetic fields(in terms of the the unit vectors $\hat{x}, \hat{y}$ and $\hat{z}$) produced in the radiation zone at distance r far from the circular orbit($r \gg R$).
(b) Find the time-averaged power radiated per unit solid angle in the (θ, φ) direction.
(c)Explain what would happen if $\alpha = 0$ or $\alpha = \pi$.

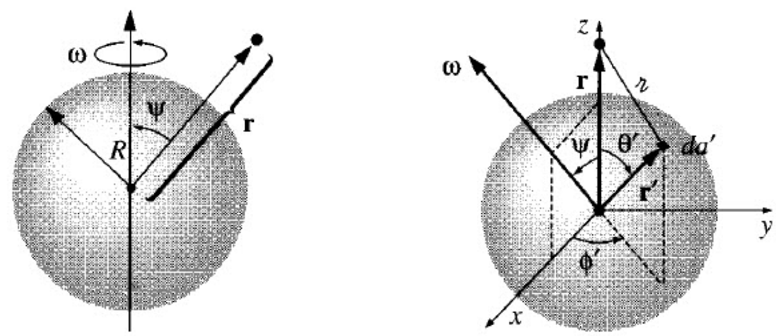




6. (5+5+5%)A specific charge density $\sigma(\theta) = k(c\cos^2\theta - 1)$ is glued over the surface of a spherical shell of radius R.

(a) Find the resulting electrostatic potential V inside and outside the sphere.

Now, assume that the sphere has a uniform surface charge distribution, σ, and spins with an angular velocity ω.
(b) Determine the vector potential $\vec{A}(\vec{r})$ inside and outside the sphere. [hint: It is easier to orient your sphere so that $\vec{r}$ lies on the z-axis and ω is tilted by angle Ψ in the xz-plane.]



(c) Demonstrate that the magnetic field inside the shell is uniform.