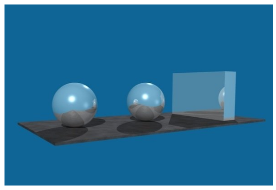
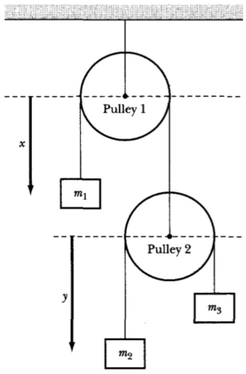
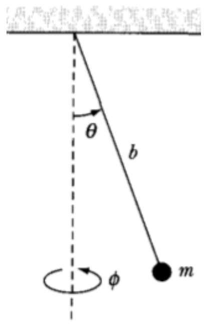
Qualifying Examination – Classical Mechanics

Spring, 2019

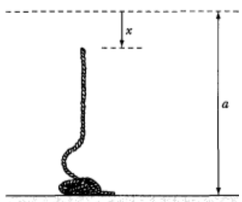


1. (The Pi machine[1], bonus 10 points) A moving mass M hits a lesser mass m initially at rest on a smooth floor. Afterwards, the latter bounces from a wall and collides with M again. This process repeats until M changes direction and m fails to catch up on it. How will you go about proving that the number of collisions[2], counting both against the wall and with M, will approach $\pi\sqrt{M/m}$ as $M \gg m$?
2. (Newtonian mechanics) A particle is projected vertically upward in a constant gravitational field with an initial speed v_0. If it experiences a resisting force kmv^2, find the speed of the particle when it returns to the initial position.
3. (Oscillation) Find the response of a damped harmonic oscillator, originally at rest in equilibrium position, to a periodic force $F(t) = \begin{cases} 0, & -2\pi/\omega < t < 0 \\ F_0, & 0 < t < 2\pi/\omega \end{cases}$.
4. (Gravitation) Show that the gravitational self-energy of a uniform sphere of mass M and radius R is $U = -(3/5)\, GM^2/R$.





5. (Lagrangian dynamics) Use Lagrangian dynamics to find the equations of motion for the double pulley in Fig.2.
6. (Hamiltonian dynamics) Use Hamiltonian to find the equation of motion for the spherical pendulum in Fig.3.
7. (Central-force motion) A mass-m rocket is fired horizontally from height h and eventually returns to Earth. Besides parabola, what are the conditions for it to assume a hyperbolic, elliptic, or circular orbit?
8. (Dynamics of a system of particles)



   (a) Consider a rope of mass per unit length ρ and length a suspended above a table in Fig.4. Find the force on the table when a length x of the rope has dropped to the table.

   (b) A rocket leaves Earth vertically under gravity. The exhaust velocity is u, and the constant fuel burn rate is α. Let the initial/final mass be m_0/m_f. Calculate the final altitude and speed of the rocket.
9. (Special theory of relativity)

   (a) What is the minimum proton energy needed to produce an antiproton $\bar{p}$ by the reaction $p + p \to p + p + (p + \bar{p})$ where the target proton is initially at rest?

   (b) Two spaceships of proper length $L_{1,2}$ approach each other with speeds $v_{1,2}$. Find the time it takes for them to pass each other as observed in the rest frame and by the two pilots.

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[1] A New York Times blog posts about this problem https://wordplay.blogs.nytimes.com/2014/03/10/pi/

[2] Click https://www.youtube.com/watch?v=jsYwFizhncE&feature=youtu.be for a video explanation.

Qualifying Examination – Statistical Mechanics

Spring, 2019

1. (The entropy of mixing, 20%) Consider two entropy functions of the ideal gas : $S = kN\ln\left[(V/N)(CE/N)^{3/2}\right]$ and $\tilde{S} = kN\ln\left[V(CE/N)^{3/2}\right]$ where C is a constant. Consider two idea gases with N_1 and N_2 particles respectively, kept in two separate volumes V_1 and V_2 at the same temperature and same density. Use S and $\tilde{S}$ respectively to find the change in the entropy of the combined system after the gases are allowed to mix in a volume $V = V_1 + V_2$ for the case of (a) two different kinds of ideal gases. (b) same kind of ideal gas. (c) Determine which one is the proper entropy function.
2. (Fermion statistics, 15%) Consider L degenerate states with energy ϵ, of which N are occupied by the fermions. (a) In the micro-canonical ensemble, find the entropy of the system. (b) By comparing to $dE = TdS + \mu dN$, find the average population of the state $f \equiv N/L$ as a function of ϵ, μ, T.
3. (Ising model, 15%) Consider the Ising model $H = -J\sum_{\langle i,j\rangle}\sigma_i\sigma_j$ where $\sigma_j = \pm 1$, and $\langle i,j\rangle$ denotes the nearest-neighbor pairs of sites. (a) For a system of coordinate number z, use mean-field approximation to find the critical temperature T_c below which spontaneous magnetization exists. (b) Show that the magnetic susceptibility $\chi \propto 1/(T - T_c)$ at $T \gg T_c$. (c) Use entropy argument to show that in 1D there is no phase transition, i.e., $T_c = 0$.
4. (1D random walk, 15%) Consider 1D random walk with equal probability going to the right and left. (a) Find the probability distribution $p(n|N)$ of finding the walker at position n after N steps. You may use $N_\rightarrow$ and $N_\leftarrow$ to represent the number of steps to the right and left respectively. (b) Use Stirling's formula to show that when $N \gg n \gg 1$ the probability distribution can be approximated by the Gaussian distribution $P(n|N) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{n^2}{2\sigma^2}\right)$ and find σ^2.
5. (Two-level system, 20%) Consider a system of N distinguishable particles, which have two energy levels, $E_0 = -\mu B$ and $E_1 = \mu B$, for each particles. Here μ is magnetic moment and B is magnetic field. The particles populate the energy levels according to the classical distribution law. (a) Calculate the average energy of such system at temperature T, and (b) the specific heat of the system. (c) Calculate the magnetic susceptibility.
6. (Bose-Einstein condensation, 20%) Consider now a number conserved Bose gas with energy dispersion $\epsilon_p = C|\vec{\boldsymbol{p}}|^a$: (a) Show that there will be a Bose-Einstein condensation if $d > a$ where d is the dimension of the space. (b) Show that the critical temperature scales with the total number of particles, N, as $T_c \propto N^{a/d}$. (Hint: you do not need to do the full calculation, and may use dimensional argument for the density of states first.)

Quantum Mechanics Qualification
Spring, 2019

Problem 1 Answer the following questions *briefly*

(a) 12% Explain the following terms briefly: (i) sponatenous emission (ii) collapse of state (iii) entanglement (iv) Berry phase

(b) 4% What is the most important consequence for a system being rotational invariance?

(c) 4% What kind of particles does the Dirac equation describe?

(d) 4% What is the Aharonov-Bohm effect?

(e) 4% Find the Hermitian conjugate of $\hat{O} = x\frac{d}{dx}$ (in terms of $\hat{O}$) and $3^{|1\rangle\langle 2|}$.

Problem 2 Consider two particles of the same mass m in one dimension. Two particles are connected by a spring with spring constant k.

(a) 5% Assuming that two particles are non-identical, calculate the eigen-energies of such a system with a total momentum p.

(b) 7% Following (a), suppose that two particles carry charges of q and $-q$ and move along x axis. A uniform electric field E is applied along $+x$ direction. Assuming the Coulomb interaction between these two charges is negligible, find the root-mean-square relative distance (i.e., minimum of $\sqrt{\langle (x_1 - x_2)^2 \rangle}$) between two particles when in the nth eigenstate.

(c) 10% Following (a), suppose that in addition to the action of spring, both particles are under the influence of the external potential $V(x) = \alpha x^2/2$. Assuming that two particles are identical spinless-fermions, calculate the eigen-energies of such a system.

Problem 3

(a) 5% Find $\mathbf{L} \cdot \mathbf{S}\, Y_1^{-1}(\theta, \phi)|+\rangle$ in terms of $Y_l^m(\theta, \phi)$, $|+\rangle$, and $|-\rangle$, where $\mathbf{L}$ is the orbital angular momentum vector operator and $\mathbf{S}$ is the spin vector operator.

(b) 8% Suppose that a particle has a magnetic dipole moment $\mu = g\mu_b \mathbf{J}$, where g is the g-factor, $\mu_b = e\hbar/2m$ is the Bohr magneton, and $\mathbf{J}$ is the total angular momentum.

(i) If the particle is placed in a uniform magnetic field $\mathbf{B}$ with the Hamiltonian being given by $H = -\mu \cdot \mathbf{B}$, find the equation of motion for the average total angular momentum $< \mathbf{J} >$.

(ii) Now consider a special case when $\mathbf{J} = \mathbf{S}$ with $s = 1/2$. Suppose that the magnetic field is $\mathbf{B} = B_0 \hat{z}$ with B_0 being a constant. At $t = 0$, the spin of the particle is measured to be pointing along the positive y-axis. Find $\langle S_x \rangle$ and $\langle S_z \rangle$ at $t > 0$.

(c) 7% Consider a system of two particles with spins $s_1 = 1$ and $s_2 = 1/2$. By considering the addition of two spins, find all possible eigenvalues to the operator $(\mathbf{S}_1 - 2\mathbf{S}_2)^2$.

Problem 4 15%

Consider the neutron-neutron scattering where the interaction potential is approximated by $V_0 \vec{s}_1 \cdot \vec{s}_2 e^{-\frac{r^2}{2a^2}}$, where $\vec{s}_1$ and $\vec{s}_2$ are the spin vector operators of two neutrons (spin 1/2) and $V_0 > 0$. In the lab frame, the scattering is set up in the way that one neutron is initially at rest, while the other one is incident with a momentum $\hbar k$. Both neutrons are not polarized. Use the first Born approximation to calculate the differential cross section $\sigma(\theta, \phi)$ in the center of mass (CM) frame. What is the differential cross section in the lab frame?

Problem 5 15%

Consider a particle of mass m confined in a two dimensional box, locating in the range $-a/2 \le x \le a/2$ and $-a/2 \le y \le a/2$. Inside the box, the potential that acts on the particle is $V(x, y) = -\lambda xy$ with $\lambda > 0$. Use the perturbation theory to calculate the energy shifts for the degenerate first excited states to first order of λ. What are the first order wave functions?

Electrodynamics Qualifying Examination, Feb., 2019.

You must provide the details or reasonings to justify your answers.

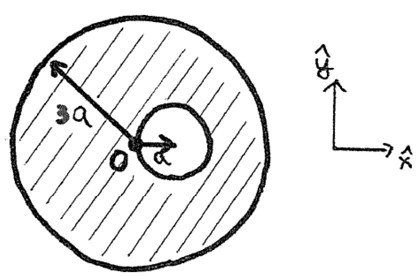
1. (5%+5%+5%) (a) What is Poynting's theorem?
   (b) Radio station Philharmonic Radio Taipei Co. radiates a power of 2kW at about 90.7 MHz from its antenna in HuKou, approximately 10km from NTHU. Obtain a rough estimate of its electric field at NTHU in volts per meter. What is the polarization of the radio wave?

2. (5%+10%)
   (a) What are Green's first identity and Green's Theorem?
   (b) A point charge is located at $(0, 0, d)$ above the grounded conducting plane at $z = 0$ which extends to infinite. Find the Green's function $G(\mathbf{x}, \mathbf{x}')$ which satisfies the Dirichlet boundary condition.

3. (10%) In the SI unit, show that $|\vec{E}|^2 - c^2|\vec{B}|^2$ is Lorentz invariant.

4. (10+10%) Consider two concentric shells of radii a and $b(b > a)$. Find the potential everywhere for the following given potentials:
   (a) $V(r = a, \theta) = 0$ and $V(b, \theta) = V_0 \sin^2 \theta$.
   (b) $V(r = a, \theta) = V_0 \cos \theta$ and $V(b, \theta) = -V_0$.
   (Hint: the Legendre polynomials $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = (3x^2 - 1)/2$.)

5. (10%) An infinitely long circular cylinder of radius $3a$ with an infinitely long cylindrical hole of radius a is displaced so that its center is at a distance a from the center of the big cylinder. The solid part of the cylinder carries a current I, distributed uniformly over the cross-section and pointing in $\hat{z}$. Assume that $\mu = \mu_0$, determine the magnetic field, $\vec{B}$, throughout the hole.



6. (5%+10%)(a) What is gauge transformation?
   (b) Construct one set of the scalar and vector potentials in Coulomb gauge for a point charge q sitting at the origin in a uniform $\vec{B} = B_0\hat{z}$.

7. (5%+10%) (a) What is Maxwell's stress tensor?
   (b) Use Maxwell's stress tensor to calculate the force experienced by a point charge q in a uniform electric field $\vec{E} = E_0\hat{z}$.