Quantum Mechanics Qualification
Spring, 2017

Problem 1 Answer the following questions *briefly*

(a) 20% Explain the following terms briefly: (i) sponatenous emission (ii) Hilbert space (iii) WKB approximation (iv) entangled states (v) Berry phase

(b) 5% What are generators for the translations in space?

(c) 5% Let M_{xy} be the mirror imaging operator with respect to x-y plane. Find how the total angular momentum, J_x, J_y, and J_z change under M_{xy}.

Problem 2 Consider a harmonic oscillator characterized by the Hamiltonian

$$H = 4a^\dagger a + 2 \tag{1}$$

where $a = (q + ip)/\sqrt{2}$, $a^\dagger = (q - ip)/\sqrt{2}$,and $[q, p] = i$.

(a) 5% If the harmonic oscillator is under the perturbation $V = a + a^\dagger$, find the eigen-energies of the system.

(b) 10% Instead of the perturbation given in (a), the perturbation is replaced by $V = a^2 + (a^\dagger)^2$. What are the eigen-energies of this system? What is the ground state wave function? Remember to fix the normalization.

Problem 3 A particle with charge $\mathbf{q}$ and mass m moves in the presence of a magnetic field $\mathbf{B}(\mathbf{r}, t)$ and an electric field $\mathbf{E}(\mathbf{r}, t)$. Let the vector and scalar potentials that describe $\mathbf{B}$ and $\mathbf{E}$ be $\mathbf{A}(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$.

(a) 5 % Denote the velocity operator by $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ with $\mathbf{x}$ being the position operator in the Heisenburg picture.

(i) Express the canonical momemtun operator $\mathbf{p}$ in terms of $\mathbf{v}$ and $\mathbf{A}$.

(ii) Using (i), find $\mathbf{v} \times \mathbf{v}$ in terms of $\mathbf{B}$.

(b) 6% Using results of (a), show that

$$m\frac{d^2\mathbf{x}}{dt^2} = q\left[\mathbf{E} + \frac{1}{2c}\left(\frac{d\mathbf{x}}{dt} \times \mathbf{B} - \mathbf{B} \times \frac{d\mathbf{x}}{dt}\right)\right],$$

which is the quantum-mechanical version of the Lorentz force.

(c) 12% Consider a special case when $\mathbf{E}(\mathbf{r}, t) = 0$ and the magnetic field is uniform and is pointing to the z-direction with $\mathbf{B} = B\hat{z}$. By comparing the Hamiltonian and results obtained in (a) with those of the 1D simple harmonic oscillator problem, show that the energy eigenvalues are

$$E_(k, n) = (\hbar^2 k^2)/2m + ((|eB|\hbar)/mc)(n + 1/2) \tag{2}$$

where $\hbar k$ is the eigenvalue to the momentum $\hat{p}_z$ operator and n is a positive integer (including zero).

Problem 4

(a) 12% Work out the Clebsch-Gordan (CG) coefficients of $1/2 \otimes 1/2 \otimes 1/2 = 3/2 \oplus 1/2 \oplus 1/2$.

(b) 8 % Using results of (a), consider a system of two particles with spins $s_1 = 3/2$ and $s_2 = 1/2$. The Hamiltonian of these two particles is given by $H = J\vec{S}_1 \cdot \vec{S}_2$, with J being a given constant.

(i) If we perform measurement on the total energy of the system, find all possible values of energy we can get.

(ii) At $t = 0$, the system is in the simutaneous eignestate of S_1^2, S_2^2, S_{1z}, and S_{2z}: $|\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle$. Evaluate the probability of finding the system in the state $|\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\rangle$ at $t > 0$.

Problem 5 12%

Consider the mutual elastic scattering of two spin-1/2 fermions. The Hamiltonian for this system is $H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(\vec{r}_1 - \vec{r}_2)$, where $V(\vec{r}) = V_0 e^{-\frac{r^2}{2a^2}}$. In the lab frame, the scattering is set up in the way that one fermion is initially at rest, while the other one is incident with a momentum $\hbar k$. Both fermions are not polarized. Use the Born approximation to calculate the differential cross section $\sigma(\theta, \phi)$ in the center of mass (CM) frame. What is the differential cross section in the lab frame? What would be the differential cross section in the CM frame if these particles are identical fermions?

Qualifying Examination – Statistical Mechanics

Spring, 2017

1. (Terminologies, 16 points) Please explain in detail the following quantities:
   (a) Bose-Einstein condensation
   (b) Correlation function and correlation length
   (c) Debye model versus Einstein model
   (d) Fluctuation-dissipation theorem

2. (Fermi gas, 30 points) For N ideal electrons of individual mass m in volume V,
   (a) Derive the Fermi energy E_F.
   (b) Derive the degenerate pressure at zero temperature. How will your result change if N/V is high enough, as in dwarf stars, to become relativistic?
   (c) Show that $-\frac{\mathrm{d}n_{\mathrm{FD}}(\varepsilon)}{\mathrm{d}\varepsilon}$, where $n_{\mathrm{FD}}(\varepsilon) = \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)}+1}$ denotes the Fermi-Dirac distribution and $\beta \equiv \frac{1}{k_{\mathrm{B}}T}$, can be treated as a probability distribution $P(\varepsilon)$ that obeys: (i) $\int_0^\infty P\,\mathrm{d}\varepsilon = 1$, (ii) $\int_0^\infty (\varepsilon-\mu) P\,\mathrm{d}\varepsilon = 0$, (iii) $\int_0^\infty (\varepsilon-\mu)^2 P\,\mathrm{d}\varepsilon \approx (k_{\mathrm{B}}T)^2$ at temperature $T \ll E_F$.
   (d) Derive the expression for chemical potential μ at temperature $T \ll E_F$.
   (e) Derive the expression for specific heat C_V at temperature $T \ll E_F$.

3. (Ising model, 20 points) The Ising Hamiltonian looks like $H = -J\sum_{\langle i,j\rangle} S_i S_j$ where $\langle i,j\rangle$ denotes sites i and j are nearest neighbors and $S = \pm 1$. Use q to denote the number of nearest neighbors for each spin.
   (a) Derive the expression for Curie temperature T_C in the mean field approximation (Hint: T_C refers to the temperature below which spins become parallel aligned.)
   (b) Solve the Ising model in one dimension exactly and show that $T_C = 0$, i.e., no phase transition to a magnetic state at any temperature for an Ising chain.

4. (Ideal and non-ideal classical gas, 24 points)
   (a) Find the Helmholtz free energy F for N ideal classical particles of individual mass m in volume V and at temperature T. (Note: Remember to include $\frac{1}{N!}$ to prevent the Gibbs paradox. Given the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$)
   (b) Derive the state equation $PV=Nk_{\mathrm{B}}T$ by $P = -\left.\frac{\partial F}{\partial V}\right|_{T,N}$.

(c) Consider the fact that real particles (i) repel each other at close range with a hard volume v and (ii) attract each other at long range by adding a potential energy term[1] $-\alpha\left(\frac{N}{V}\right)^2 V$ to the free energy. Show that these modifications lead us to the Van der Waals equation: $\left[P+\alpha\left(\frac{N}{V}\right)^2\right](V-Nv)=Nk_BT$.

(d) Take two containers of different gas and open the partition to allow them to mix. What is the entropy change after the mixing? But, had the two gases been the same, we do not expect any entropy change! What happened?

5. (Path integral and partition function, 20 points)
Please proceed with the following steps to introduce yourself to the concept of (Feynman's) path integral and how it can become useful in statistical mechanics. Take the infinite potential well, $V(x)=\begin{cases}0, & \text{for } 0\le x\le L\\ \infty, & \text{elsewhere}\end{cases}$, as an example.

(a) Find all classical paths that can bring an electron from x_i to x_f within time τ where $0\le x_{i,f}\le L$.

(b) Sum up their contributions to the propagator, $\mathrm{G}(x_i,0;\ x_f,\tau)=\sqrt{\frac{-im}{2\tau\pi}}\sum_{\text{path}}(-1)^n e^{iS}$, where S denotes the classical action, $\int_0^\tau\left(\frac{m}{2}v^2-V\right)dt$, and n is the number of times the electron gets bounced from the walls. The coefficient, $\sqrt{\frac{-im}{2\tau\pi}}$, is to insure $\lim_{\tau\to 0}\mathrm{G}(x_i,0;\ x_f,\tau)=\delta(x_i-x_f)$ – treat it as given.

(c) Use the Poisson summation formula, $\sum_{n=-\infty}^{\infty}\tilde{f}\,(kn)=\frac{2\pi}{k}\sum_{\ell=-\infty}^{\infty}f\left(\frac{2\ell\pi}{k}\right)$ where $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)e^{ikx}dk$ is the inverse Fourier transformation, to rewrite the summation in (b).

(d) Try to reduce $\sum_{\ell=-\infty}^{\infty}(\text{something})$ to the Schrodinger form of propagator, i.e. $\sum_{\ell=1}^{\infty}\psi^*{}_\ell(x_i)\psi_\ell(x_f)\,e^{-i\varepsilon_\ell\tau}$, and identify the eigenfunction $\psi_\ell(x)$ and eigenvalue ε_ℓ. The knowledge of ε_ℓ will now allow you to write down the partition function $Z=\sum_{\ell=1}^{\infty}e^{-\beta\varepsilon_\ell}$ for a quantum-mechanical electron.

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[1] The parameter, α, describes the strength of pairwise attractive force between particles, while $\left(\frac{N}{V}\right)^2$ comes from the joint probability. Finally, the V factor comes from integrating the energy density $-\alpha\left(\frac{N}{V}\right)^2$ over the whole volume to obtain the potential energy.

Qualification Exam for PhD Candidates (Classical Mechanics, spring 2017)

1. (10%) What is the total cross section for the elastic scattering of a beam of particles of radius r from a fixed solid sphere with radius R?

2. (15%) The corners of a rectangle lie on the ellipse $(x/a)^2 + (y/b)^2 = 1$. Where should the corners be located in order to maximize the area of the rectangle?

3. (15%) A photon hits an electron at rest, producing an electron-positron pair:
$$\gamma + e^- \to e^- + e^+ + e^-.$$
Use special relativity to calculate the minimum energy of the incident electron.

4. (15%) Consider a damped oscillator. An external driving force is applied to the oscillator. The equation of motion is
$$d^2x/dt^2 + 2\beta dx/dt + \omega_0^2 x = A\cos(\omega t).$$
Determine the particular solution of this solution.

5. (15%) At the origin $x = y = 0$ a machine can shoot a ball with constant speed v_0 at any angle. Find the boundary of the region which can be reached by the ball.

6. (15%) Find Hamilton's equations of motion for an unharmonic oscillator for a particle of mass m and the potential energy is given by
$$V(x) = \frac{k}{2}x^2 + \frac{a}{4}x^4$$
where k and a are constants. (You are not required to solve these equations).

7. (15%) The Hamiltonian of a harmonic oscillator is
$$H = \frac{p^2 + m^2\omega^2 q^2}{2m} = E.$$
(a) Write down the Hamilton-Jacobi equation (do not solve it). (b) Use the method of action-angle variables to calculate the frequency of oscillation.

Electrodynamics Qualification, Feb., 2017.

You must provide the details or reasonings to justify your answers.

1. (5% each) Concisely explain the following terms:
   (a) Why the sky is blue in the noon of a sunny day and red at sunset.
   (b) Liénard-Wiechert potentials.
   (c) Plasma frequency.
   (d) Perfect conductor and superconductor, how they differ from each other.
   (e) Gauge transformation.
   (f) Skin depth.
   (g) Synchrotron radiation.
   (h) Linear and circular polarization of light.

2. (10%) Two concentric shells of inner and outer radii a and $b(b > a)$, respectively. The inner shell is grounded with $V(r = a, \theta) = 0$. If $V(b, \theta) = V_0 \cos^2 \theta$, find the potential everywhere. (Hint: the Legendre polynomials $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = (3x^2 - 1)/2$.)

3. (15%) An infinite long circular cylinder of radius $3a$ with an infinite long cylindrical hole of radius a is displaced so that its center is at a distance a from the center of the big cylinder, see Fig.1. The solid part of the cylinder carries a current I, distributed uniformly over the cross section and pointing out the plane of the paper. Assume that $\mu = \mu_0$, determine the magnetic field, $\vec{B}$, throughout the hole.

Fig.1

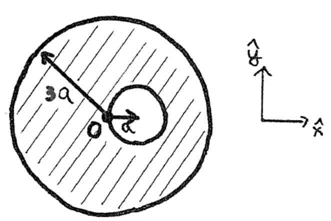
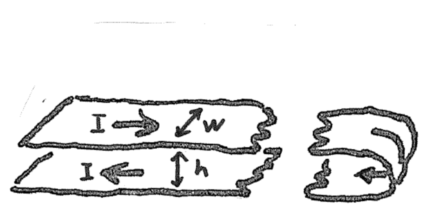


Fig.2



4. (5+5+5%) A transmission line is constructed from two thin metal "ribbons", of width w, a very small distance $h \ll w$ apart, Fig.2. The current travels down one trip and back along the other. In each case, it spreads out uniformly over the surface of the ribbons.
   (a) Find the capacitance per unit length, $\mathcal{C}$.
   (b) Find the inductance per unit length, $\mathcal{L}$.
   (c) What is the product $\mathcal{LC}$, numerically? What does it mean?(Also specify the unit carefully.)

5. (10%) Radio station Philharmonic Radio Taipei Co. radiates a power of 2kW at about 90.7 MHz from its antenna in HuKou, approximately 10km from NTHU. Obtain a rough estimate of its electric field at NTHU in volts per meter.

6. (10%) Suppose the yz plane forms the boundary between two linear media with $\mu = \mu_0, \epsilon_1$ for $x < 0$ and $\mu = \mu_0, \epsilon_2$ for $x > 0$. A plane wave of frequency ω, traveling in the x direction and polarized in the y direction, approaches the interface from $x = -\infty$. Derive the transmission coefficient and express it in terms of the reflection indices n_1 and n_2.