Electrodynamics Qualifying Exams

February 2023

You must provide the details or reasonings to justify your answers.

Problem 1: Simple Questions (5% each = 35% total)

1. Write down the four Maxwell's equations. Specify all the numerical values of the relevant physical constants and the units you adopt.
2. Why are electromagnetic waves transverse in the vacuum?
3. In one particular inertial frame, the electric and magnetic fields are measured as $\vec{E} = (9, -1, 7)$ and $c\vec{B} = (5, 0, -2)$ in some units, respectively. In a different frame, $\vec{E} = (2, 0, 4)$. Give one possible $c\vec{B}'$.
4. Explain the terms Lorentz gauge and Coulomb gauge.
5. Explain the term Green's function.
6. Explain the terms Dirichlet and Neumann boundary conditions.
7. How many real degrees of freedom does the quadrupole tensor have? Explain your answer.

Problem 2: Simple derivations (10% each = 20% total)

1. Derive the conservation of charge from the Maxwell equations.
2. Derive the wave equation for electromagnetic fields from the Maxwell equations.

Problem 3: Moving charge (15%)

What are a point charge's electric and magnetic fields at a constant velocity $\vec{v}$?

Problem 4: Image Charges (18%)

Consider the volume $V = \{\vec{r} : 0 \leq x \leq \infty, 0 \leq y \leq \infty, -\infty \leq z \leq \infty\}$ which is limited by grounded metal plates. On the surface of the metal plates (at $x = 0$ and $y = 0$) and at infinity the potential vanishes. In the volume there is a point charge q at $\vec{r}_0$.

1. (5%) Find the electric potential $\phi(\vec{r})$ in V and check that it fulfills the boundary conditions.

2. (8%) Calculate the surface charge density

$$\sigma = -\epsilon_0 \, \hat{n} \cdot \vec{\nabla} \phi \, , \tag{1}$$

with $\hat{n}$ the normal vector on the surface S.

3. (5%) Calculate the total charge

$$Q = \int \sigma \, \mathrm{d} \, S \tag{2}$$

on the plates. You can use Gauss' theorem to calculate Q.

Problem 6: Multipole Moments (12%)

Consider two homogeneously charged, solid hemispheres with radius R which are separated in the x-y-plane by a negligible slit. The upper hemisphere shall have total charge $+Q$ and the lower hemisphere shall have total charge $-Q$.

1. (3%) Write down an expression for the charge density, ϱ, of the two hemispheres.

2. (4%) Calculate either the spherical or the Cartesian monopole moment.

3. (5%) Calculate either the spherical or the Cartesian dipole moment.

Quantum Mechanics Spring 2023
Qualifying Exam

You must show your work. No credits will be given if you don't show how you get your answers.
Boldface characters like $\mathbf{v}$ refer to vectors ($= \vec{v}$).

You may use the following formula:

- The Schrödinger equation:

$$i\hbar \frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(t)\rangle$$

where $\hat{H}$ is the Hamiltonian $\dfrac{\hat{\mathbf{p}}^2}{2\mu} + V(\hat{\mathbf{r}})$ and μ is the mass of the particle.

For energy eigenstates, this reduces to the time-independent Schrödinger equation (in spherical coordinates)

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\psi(\mathbf{r}) + \langle\mathbf{r}|\,\frac{\hat{\mathbf{L}}^2}{2\mu r^2}\,|\psi\rangle + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

$\hat{\mathbf{L}}$ is the orbital angular momentum operator, $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$.

- For a 1-D simple harmonic oscillator (SHO), $\hat{H} = \dfrac{\hat{p_x}^2}{2m} + \dfrac{1}{2}m\omega^2\hat{x}^2$:

The raising and lowering operators are

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p_x}\right), \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p_x}\right)$$

and $[\hat{a}, \hat{a}^\dagger] = 1$. The operators get their names from the facts that

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,,$$

where $|n\rangle$'s are the energy eigenstates of the 1D SHO.

- The Pauli matrices:

$$\sigma_{\mathrm{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_{\mathrm{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_{\mathrm{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• Time-independent Perturbation: Consider $\hat{H} = \hat{H}_0 + \hat{H}_1$, where $\hat{H}_1$ is a small perturbation. The first order energy correction to the n-th energy eigenvalue of $\hat{H}_0$ is

$$E_n^{(1)} = \left\langle n^{(0)} \middle| \hat{H}_1 \middle| n^{(0)} \right\rangle$$

where $\left|n^{(0)}\right\rangle$ is the n-th eigenstate of $\hat{H}_0$.

The second order energy correction is

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\left\langle n^{(0)}\right| \hat{H}_1 \left|m^{(0)}\right\rangle|^2}{E_0^n - E_0^m}.$$

• Time-dependent Perturbation: Consider $\hat{H} = \hat{H}_0 + \hat{H}_1(t)$, where $\hat{H}_1(t)$ is a small perturbation. The state $|\psi(t)\rangle$ can be expanded in terms of the unperturbed eigenstates $\left|n^{(0)}\right\rangle$:

$$|\psi(t)\rangle = \sum_{n=0} d_n(t) e^{-i\frac{E_n^{(0)} t}{\hbar}} \left|n^{(0)}\right\rangle.$$

If initially ($t = 0$) the particle is at state $\left|i^{(0)}\right\rangle$, the transition amplitude for the particle to be in state $\left|f^{(0)}\right\rangle$ at time t is

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar} \int_0^t dt' \left\langle f^{(0)} \right| \hat{H}_1(t') \left| i^{(0)} \right\rangle e^{i\frac{(E_f^{(0)} - E_i^{(0)})}{\hbar} t'}.$$

• 3-D Scattering: The wave function $\psi(\mathbf{r})$ of a particle scattering off a potential $V(\mathbf{r})$ has the asymptotic behavior

$$\psi(\mathbf{r}) = Ae^{ikz} + Af(\theta, \phi)\frac{e^{ikr}}{r}, \quad r \to \infty,$$

where the incident wave is in the z-direction.

• Born Approximation: Useful for high-energy scattering.

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int d^3\mathbf{r}' V(\mathbf{r}') e^{-i\mathbf{q}\cdot\mathbf{r}'},$$

where m is the mass of the scattering particle, and $\hbar\mathbf{q} = \hbar\mathbf{k}_f - \hbar\mathbf{k}_i$ is the momentum transferred from the initial state $e^{i\mathbf{k}_i\cdot\mathbf{r}}$ to the final state $e^{i\mathbf{k}_f\cdot\mathbf{r}}$.

• Partial wave expansion: Useful for low-energy scattering. For spherically-symmetric $V(r)$,

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) \frac{e^{i\delta_l}}{k} \sin\delta_l \; P_l(\cos\theta),$$

where δ_l is the phase shift of the l-th partial wave, and $P_l(\cos\theta)$ is the Legendre polynomial, which relates to the spherical harmonics $Y_{l,m}(\theta,\phi)$ as $Y_{l,0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$.

- Useful Gaussian integrals:

$$\int_{-\infty}^{\infty} \exp\left(-ax^2 + bx\right) dx = \exp\left(\frac{b^2}{4a}\right)\sqrt{\frac{\pi}{a}}$$

1. Consider a massive charged spin-1/2 particle placed under an external magnetic field, $\mathbf{B} = B_0\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. (5 points each)

   (a) Explain why the Hamiltonian of the system is of the form $\hat{H} = \omega_0\hat{\mathbf{S}}\cdot\hat{\mathbf{n}}$, where $\hat{\mathbf{S}}$ is the spin operator and ω_0 has the unit of angular frequency.

   (b) At $t = 0$, the particle is in the state of spin-up along the z-axis, $|\psi(0)\rangle = |+\mathbf{z}\rangle$. Write down the expectation value of energy $\langle E\rangle$ of the particle as a function of time.

   (c) If $\mathbf{n}$ points along the positive x-axis, find out the probability of the particle in the state $|+\mathbf{z}\rangle$ as a function of time.

2. Consider a particle with charge q, mass m, and placed under a 1D simple harmonic oscillator (SHO) potential. (5 points each)

   (a) Prove that the wave function $\psi_0(x)$ of the ground state $|0\rangle$ (i.e. $\psi_0(x) = \langle x|0\rangle$) of the system is a Gaussian function.

   (b) Prove that $\psi_n(x) = \langle x|n\rangle$ is an even/odd function if n is even/odd. [*Hint:* Show that the parity of $|n\rangle$ is $(-1)^n$.]

   (c) Now we turn on a constant electric field $\mathbf{E} = E_0\hat{\imath}$. Write down the resulting Hamiltonian.

   (d) Treat the electric field as a perturbation, find out the first and second order energy corrections to the n-th energy eigenvalue of the unperturbed SHO.

   (e) Explicitly solve for the energy eigenvalues of the full Hamiltonian, and prove that all higher order (beyond the second order) energy corrections vanish.

   (f) Suppose the electric field decays in time as $\mathbf{E} = E_0\exp\left(-\dfrac{t}{\tau}\right)\hat{\imath}$. At $t = 0$ the system is in the ground state. Find the transition probability to $|n\rangle$ (for all $n > 0$) as $t \to \infty$.

3. Consider a particle with mass μ confined in a 3D infinite potential well, $V(\mathbf{r}) = 0$ for $|\mathbf{r}| < a$, ∞ or $|\mathbf{r}| > a$.

   (a) Explain why the energy eigenstates are of the form $\psi_E(r,\theta,\phi) = R_{n,l}(r)Y_{l,m}(\theta,\phi)$, where n, l, m are quantum numbers associated with energy, total angular momentum, and angular momentum along the z-axis. (5 points)

   (b) Write down the Schrödinger equation for an energy eigenstate $\psi_E(r,\theta,\phi)$ for the case $l = 0$, along with proper boundary conditions. (5 points)

(c) Rewrite the radial equation in terms of $U(r) = rR(r)$ for the case $l = 0$, and solve for the corresponding energy eigenvalues. (10 points)

4. 3D Scattering: Consider a particle of mass m moving in 3D.

   (a) Let $\psi(\mathbf{r}, t)$ be the wave function associated with the particle. Prove that the probability current density $\mathbf{j}$ is $\frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)$ by taking the time derivative of the probability density. What is the unit of $\mathbf{j}$? (5 points)

   (b) The particle is scattering off a potential $V(\mathbf{r})$. Prove that the differential cross section $\frac{d\sigma}{d\Omega}$ is related to the scattering amplitude $f(\theta, \phi)$ as
   $$\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2$$
   if the particle approaches the potential along the positive z-direction. (5 points)

   (c) If the potential is of the form $V(\mathbf{r}) = V_0 \exp\left(-r^2/R^2\right)$ (R is a constant), explain why $f(\theta, \phi)$ depends only on θ. (5 points)

   (d) Use the Born approximation, calculate the the scattering amplitude $f(\theta)$ for the potential $V(\mathbf{r}) = V_0 \exp\left(-r^2/R^2\right)$. (10 points) [*Note:* You should express your answer in terms of $k(= |\mathbf{k}_i|)$ and θ.]

   (e) Derive the s-wave scattering length in the limit of zero energy for the potential $V(\mathbf{r}) = V_0 \exp\left(-r^2/R^2\right)$.(10 points)

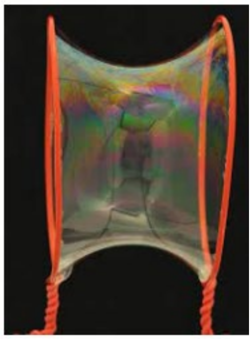
Qualification Exam. Problem Set

Classical Mechanics

Spring, 2023

1. A point mass is constrained to move on a massless hoop of radius a fixed in a vertical plane that rotated about its vertical symmetry axis with constant angular speed ω.
   (a)、(5 points) Obtain the Lagrange equations of motion assuming the only external forces arise from gravity and find the constants of motion.
   (b)、 (10 points) Show that if ω is greater than a critical value ω_0, there can be a solution in which the particle remains stationary on the hoop at a point other than at the bottom, but if $\omega < \omega_0$, the only stationary point for the particle is at the bottom.
   (c)、(10 points) Find the critical value ω_0, and the corresponding stationary point.

2. A soap film is built between two circular wires of radius R that are separated by a distance L. Air is allowed to flow through the wires. Please
   (a)、(10 points) find the shape of film.
   (b)、 (10 points) show that there exists a upper bound for L beyond which the film becomes unstable.



3. (20 points)Derive Kepler's third law: $\tau^2 \propto a^3$ where τ is the period and a the semi-major axis (長軸半長) by the following steps:
   (a)、draw an elliptic orbit with semi-major and minor axes a, b and put the sun on one of its focus (焦點),
   (b)、 denote the speed of planet at perihelion (近日點) and aphelion (遠日點) by v_p and v_a,
   (c)、 write down the conservations of angular momentum ℓ and mechanical energy at perihelion and aphelion,
   (d)、 given that the area of ellipse equals $ab\pi$, relate τ to ℓ.

4. (15 points) Solve the problem of the motion of a point projectile in a vertical plane, using the Hamilton-Jacobi method. Find both the equation of the trajectory and the dependence of the coordinates on time, assuming the projectile is fired off at time $t = 0$ from the origin with the velocity v_0, making an angle α with the horizontal.

5. For a one-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2}$$

(a)、(10 points)Show that there is a constant of the motion

$$D = \frac{pq}{2} - Ht$$

(b)、(10 points)As a generalization of part (a), for motion in a plane with the Hamiltonian

$$H = |\boldsymbol{p}|^n - ar^n$$

where $\boldsymbol{p}$ is the vector of the momenta conjugate to the Cartesian coordinates, show that there is a constant of the motion

$$D = \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{n} - Ht.$$

Qualifying Examination – Statistical Mechanics

Feb 18-19, 2023

Please explain the logic behind your answers.

Problem 1. *Fermions in a two-level system (15 points):* Consider a system of N independent fermions. Assume that the single-particle Hamiltonian have only two energy levels, with energy 0 and ϵ. However, the two levels have degeneracies n_0 and n_1, which are both integers.

1. For the case of $n_0 = n_1 = 3$ with $N = 3$. Find the chemical potential μ, as a function of temperature. What is the Fermi energy $\epsilon_F = \mu(T = 0)$?

2. For the case of arbitrary value of n_0 and n_1, but with $N = n_0$. Find the chemical potential μ, as a function of temperature at the low temperature limit. What is the Fermi energy $\epsilon_F = \mu(T = 0)$?

Problem 2. *Spins and vacancies on a surface and the negative temperature (20 points):* Consider a surface with N sites and a collection of non-interacting spin $\frac{1}{2}$ particles. For each site the energy $\varepsilon = 0$ if there is a vacancy and $\varepsilon = -W$ if there is a particle present, where $-W < 0$ is the binding energy.

1. Let $N_\pm$ be the number of particle with spin $\pm\frac{1}{2}$, $Q = N_+ + N_-$ be the number of spins, N_0 be the number of vacancies, $M = N_+ - N_-$ be the surface magnetization. We have $N_+ + N_- + N_0 = N = Q + N_0$. In the microcanonical ensemble, compute the entropy $S(Q, M)$. *Hint: Calculate the number of states available to the system and express* N_+, N_-, N_0 *with* Q *and* M.

2. Let $q = \frac{Q}{N}$ be the dimensionless particle density and $m = \frac{M}{N}$ be the dimensionless magnetization density. Assuming that we are at the thermodynamic limit where N, Q and M all tend to infinity, but with q and m finite. Find the temperature $T(q, m)$. *Hint: Use Stirling's formula* $\ln N! \approx N \ln N - N$.

3. Show explicitly that T can be negative for this system. What does negative T mean? What physical degrees of freedom have been left out that would avoid this strange property?

Problem 3. *Universality beyond the ideal gas model (25 points):* The ideal gas model gives the state equation $P\frac{V}{N} = k_B T$. However, the model is pretty artificial that it consider particles as mathematical points. The Van der Waals equation take into the account that

- The particles are not mathematical points, but with some intrinsic volume. This leads to

$$P\left(\frac{V}{N} - b\right) = k_B T. \tag{1}$$

- The particles will collide and put pressure on the container. However, the particles will interact(electric dipole, magnetic dipole, ...) with each other and modify the expression of pressure. This leads to

$$(P - Q)\left(\frac{V}{N} - b\right) = k_B T. \tag{2}$$

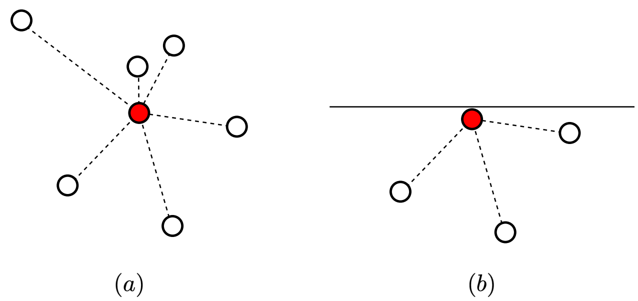


Figure 1: The toy model for interaction in the Van der Waals model. The dashed lines represent the interaction between the reference particle and its neighbors. (a) represents the case where the reference particle is deep inside the container and cannot see the boundary of the container. (b) represents the case where the reference particle is near the boundary of the container.

Let's try to use a simple averaged picture to estimate Q. The interaction between the particles are complicated. They could depends on the distance between particles as in Fig. 1(a). Let's assume the potential, U, describes the interaction between a particle deep inside the container and the rest of the particles near it. And U_W to be the potential when the reference particle is near the wall of the container. From Fig. 1(b), we estimate $U_W = \frac{U}{2}$.

1. The density deep inside the container is $\frac{N}{V}$, we expect the density near the wall of the container to be modified. A rough estimation is

$$\left.\frac{N}{V}\right|_{\text{Wall}} = \frac{N}{V} \underbrace{\mathcal{F}(U_W - U)}_{\text{modification by wall}} \overset{?}{\to} \frac{N}{V} exp\left[-\frac{1}{k_B T}(U_W - U)\right] = \frac{N}{V} exp\left[\frac{U}{2k_B T}\right] \tag{3}$$

What is the physical argument to go from the left hand side of $\overset{?}{\to}$ to the right hand side of it?

Hint: Here U should be proportional with the density of the system, so we can write $U = \frac{N}{V}u_0$ where $u_0 = \int_{B(0)} d^3\boldsymbol{r}u(r)$. Here we use $B(0)$ to denote the integral region near the reference particle and $u(r)$ is the two particle potential with r denote the distance to the reference particle. Therefore, we have

$$\left.\frac{N}{V}\right|_{Wall} = \frac{N}{V} exp\left[\frac{Nu_0}{2Vk_B T}\right]. \tag{4}$$

When $u_0 > 0 (u_0 < 0)$, we have a repulsive(attractive) potential between particles. The density near the wall will be larger(smaller) comparing with the density in the bulk. Using this picture, it is clear why we have the last equality. Since near the boundary, U_W should only include half of its neighbors, so we have $U_W = U/2$.

2. *Interaction correction to the pressure*: Assume we can apply the ideal gas equation near the wall of the container

$$P = k_B T \left(\frac{N}{V}\bigg|_{Wall} \right). \tag{5}$$

Assume we are still at the dilute gas limit. Find the expression of P to leading order correction in $bN/V \equiv bv^{-1}$ beyond the ideal gas equation by expanding the exponent. With this correction, we can modify the ideal gas model and derive the Van der Waals equation as

$$\left(P + \frac{a}{v^2}\right)(v - b) = k_B T. \tag{6}$$

Derive the expression of a using u_0.

3. Expand the Van der Waals equation, one will get a cubic equation $F_{VdW}(v) = 0$. The cubic equation has different possibilities of its solution:

   (a) three real roots of v.

   (b) one real root of v and a pair of complex root of v.

   (c) three degenerate real root $v = v_c$. This corresponds to the *critical point* of the system.

   Here, we only focus on the last case which corresponds to the case we have $F_{VdW}(v) \propto (v - v_c)^3$. Assuming the corresponding temperature and pressure in this case to be denoted as T_c and P_c. Find the expression of T_c, P_c and v_c using the parameter a, b in the Van der Waals equation.

4. Show that $\frac{P_c v_c}{k_B T_c}$ is a constant independent of the microscopic detail parameters a, b. The meaning of this expression is: for different systems, we will need different microscopic parameter a to describe how they interact and b to describe the correction due to the size of the particles. In general, a, b differs from system to system. However, at the critical point, they all satisfy this relation that the value of the product $\frac{P_c v_c}{k_B T_c}$ is the same constant.

Problem 4. *Ising model(15 points):* Considering the Ising model $H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$ where $\sigma_i = \pm 1$ and $\langle i, j \rangle$ denotes the nearest-neighbor pairs of sites.

1. For a system of coordinate number z, use mean-field approximation to find the critical temperature T_c below which spontaneous magnetization exists.

2. Show that the magnetic susceptibility $\chi \propto (T - T_c)^{-1}$ at $T \gg T_c$

3. Use entropy argument to show that in 1D there is no phase transition, i.e., $T_c = 0$.

Problem 5. *Free bosons (15 points):* Consider a number conserved Bose gas with energy dispersion $\varepsilon_p = C|\boldsymbol{p}|^\alpha$ in d dimension space:

1. Show that there will be a Bose-Einstein condensation if $d > \alpha$.

2. Show that the critical temperature scales with the total number of particles, N, as $T_c \propto N^{\alpha/d}$. *Hint: you do not need to do the full calculation and may use dimensional argument for the density of states first.*

Problem 6. *Random walk(10 points):* A drunkard is kicked out of the pub. Being fully intoxicated, he walks forward and backward with equal probability. According to the senior high school math, the probability of finding him at position n after N step is $P(n, N) = C_{N_\rightarrow}^{N} \left(\frac{1}{2}\right)^{N_\rightarrow} \left(\frac{1}{2}\right)^{N_\leftarrow}$ where $N_\rightarrow$ and $N_\leftarrow$ represent the number of steps forward and backward, respectively. By definition, we have $N = N_\rightarrow + N_\leftarrow$ and $n = N_\rightarrow - N_\leftarrow$. Assume $N \gg n \gg 1$ so that the Stirling formula can be used to approximate all large factorials: $\lim\limits_{N \gg 1} N! \approx N \ln N - N$. Show that $P(n, N)$ can be reduced to the Gaussian distribution : $P(n, N) \sim \frac{1}{\sqrt{N}} exp \left[-\frac{n^2}{4N}\right]$.